Non-Unimodularity

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UNIVERSITY OF BATH

CIRM, 24 March 2009

Numeration: Mathematics and Computer Science

β-Substitutions

- Let $\beta > 1$ be a *PV-number* (*Pisot*-number) and consider the greedy expansion to base β.
- Any (nonnegative) number x has such an expansion $\sum_{n=m}^{\infty} a_n \cdot \beta^{-n}$ with coefficients $a_i \in \{0, 1, \ldots, \lceil\beta\rceil - 1\}$.
- One might be interested in characterising the set of β-integers

$$\mathbb{Z}_\beta = \left\{ x \mid x = \sum_{n=m}^{0} a_n \cdot \beta^{-n} \right\}$$

- $\mathbb{Z}_\beta$ can also be obtained using a substitution

  There are essentially two classes of substitutions depending on the β-expansion of 1 being periodic or eventually periodic.

  $a \longrightarrow a^{d_1} b,\ b \longrightarrow a^{d_2} c,\ \ldots,\ z \longrightarrow a^{d_n}$

Earlier This Afternoon. . .

. . . we saw the polynomial $p(x) = x^4 - x^3 - x^2 - x - 1$, which has dominant (Pisot!) root $\beta \approx 1.928$.

- Alg. conj. are $\lambda_r \approx -0.775$ and $\lambda_c, \overline{\lambda}_c \approx -0.076 \pm i \cdot 0.815$.
- Associated substitution: $a \to ab$, $b \to ac$, $c \to ad$, $d \to a$.

  Geometrically, we can describe this as a tile substitution:

  Choose $\ell_a = 1$, $\ell_b = \beta - 1$, $\ell_c = \beta^2 - \beta - 1$ and $\ell_d = \beta^3 - \beta^2 - \beta - 1$ (the components of the PF-eigenvector $\underline{\ell}$ of the *substitution*/*Abelianization* matrix).

  Then inflate and subdivide, and the set of left endpoints of the intervals yields $\mathbb{Z}_\beta$.

Geometric Realization in $\mathbb{C} \times \mathbb{R}$

We now replace β^k in the expansion of $x \in \mathbb{Z}_\beta$ by $(\mathrm{Re}(\lambda_c^k), \mathrm{Im}(\lambda_c^k), \lambda_r^k)^T$ to get an object in $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$.

- Calling this the $\star$-map, the closure of $\mathbb{Z}_\beta^\star$ is now of interest. Let Ω_i is the closure of the set of i-type endpoints; then

$$\mathrm{cl}\, \mathbb{Z}_\beta^\star = \Omega_a \cup \Omega_b \cup \Omega_c \cup \Omega_d,$$

- The substitution rule yields an *iterated function system*. Using the notations

$$\lambda^\star = \begin{pmatrix} \mathrm{Re}\,\lambda_c & -\mathrm{Im}\,\lambda_c & 0 \\ \mathrm{Im}\,\lambda_c & \mathrm{Re}\,\lambda_c & 0 \\ 0 & 0 & \lambda_r \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

we get:

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$$\Omega_a = \lambda^\star \Omega_a \cup \lambda^\star \Omega_b \cup \lambda^\star \Omega_c \cup \lambda^\star \Omega_d$$

$$\Omega_b = \lambda^\star \Omega_a + t$$

$$\Omega_c = \lambda^\star \Omega_b + t$$

$$\Omega_d = \lambda^\star \Omega_c + t$$

A Measure Calculation

The sets Ω_i are compact and for their Lebesgue-measure we find:

$$\mu(\Omega_a) = \mu(\lambda^\star \Omega_a \cup \lambda^\star \Omega_b \cup \lambda^\star \Omega_c \cup \lambda^\star \Omega_d)$$

$$\mu(\Omega_j) = \mu(\lambda^\star \Omega_{j-1} + t)$$

A Measure Calculation

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The sets Ω_i are compact and for their Lebesgue-measure we find:

$$\mu\left(\Omega_a\right) \leq \mu\left(\lambda^\star \Omega_a\right) + \mu\left(\lambda^\star \Omega_b\right) + \mu\left(\lambda^\star \Omega_c\right) + \mu\left(\lambda^\star \Omega_d\right)$$

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$$\mu(\Omega_a) \leq |\lambda_c|^2 \cdot |\lambda_r| \cdot (\mu(\Omega_a) + \mu(\Omega_b) + \mu(\Omega_c) + \mu(\Omega_d))$$

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But $|\lambda_c|^2 \cdot |\lambda_r| = 1/\beta$ and we have componentwise

$$\beta \cdot \begin{pmatrix} \mu(\Omega_a) \\ \mu(\Omega_b) \\ \mu(\Omega_c) \\ \mu(\Omega_d) \end{pmatrix} \leq \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mu(\Omega_a) \\ \mu(\Omega_b) \\ \mu(\Omega_c) \\ \mu(\Omega_d) \end{pmatrix}$$

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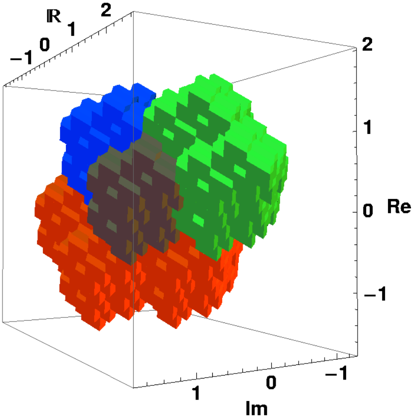

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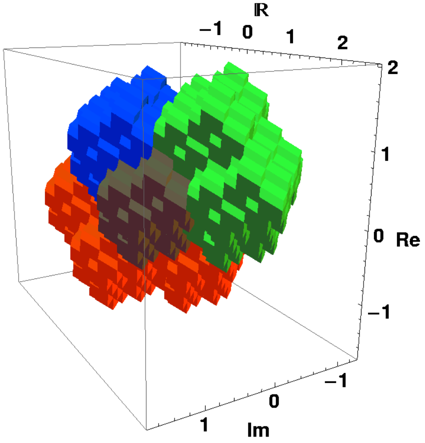

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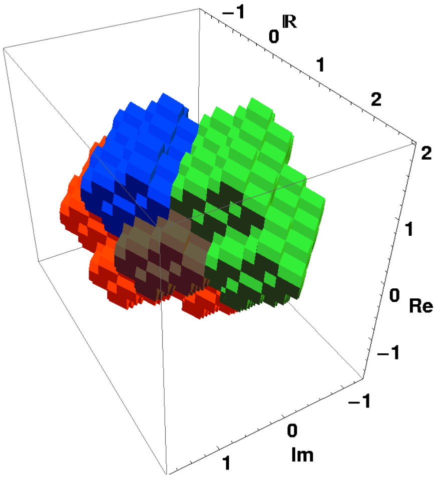

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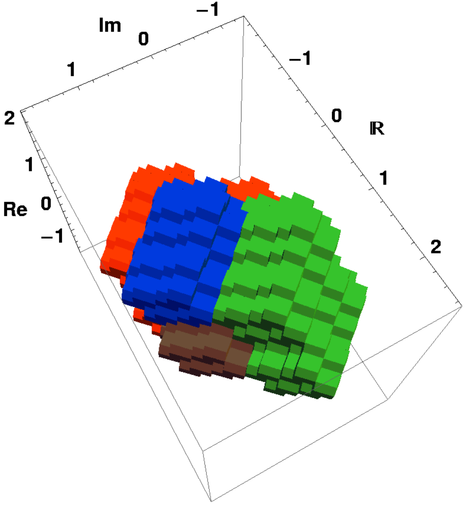

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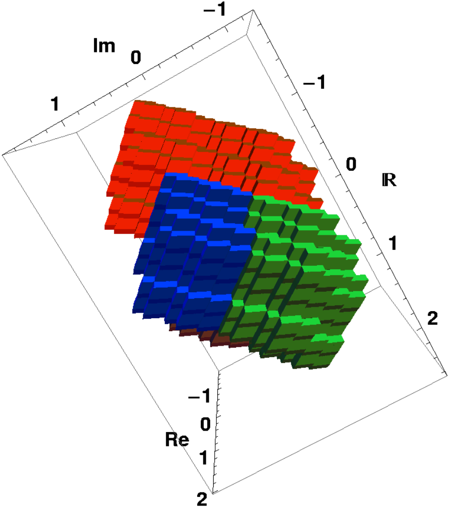

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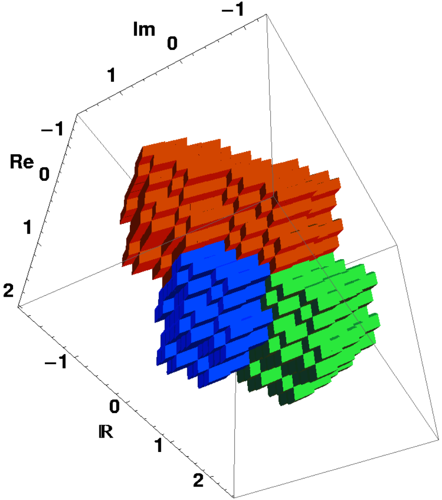

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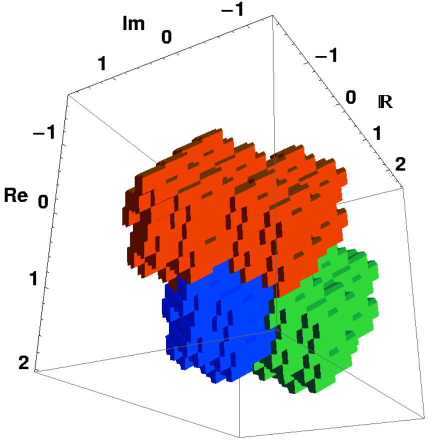

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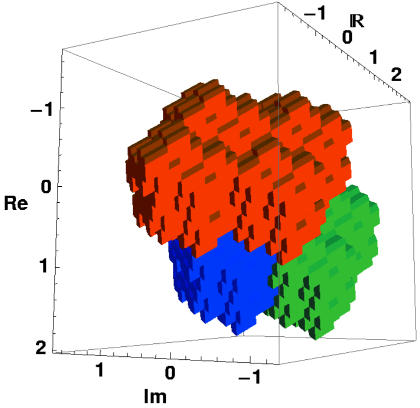

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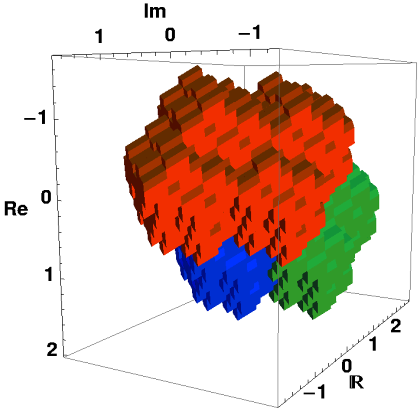

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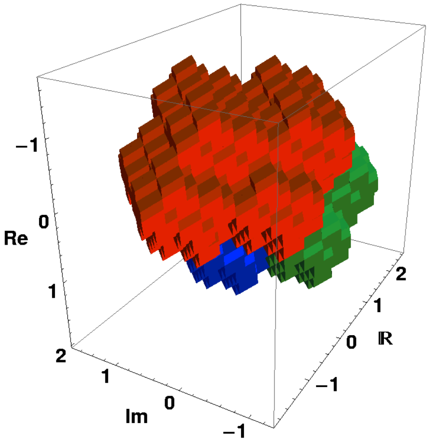

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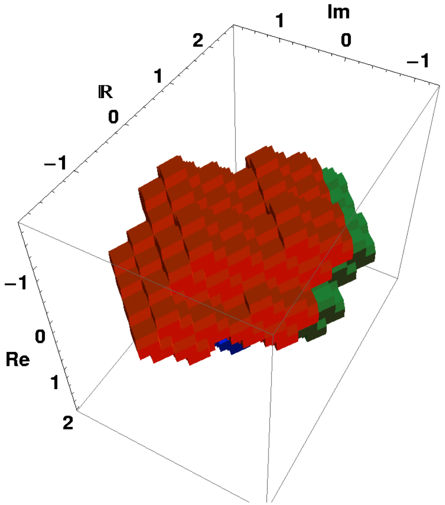

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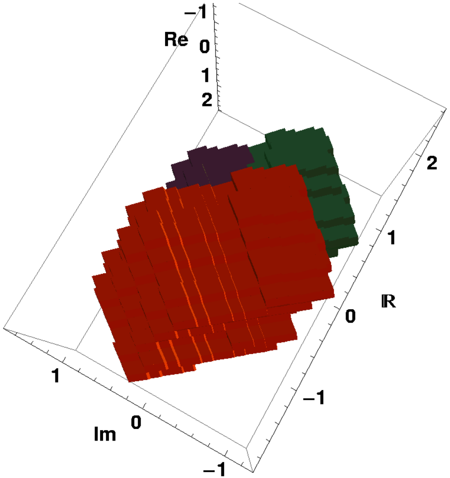

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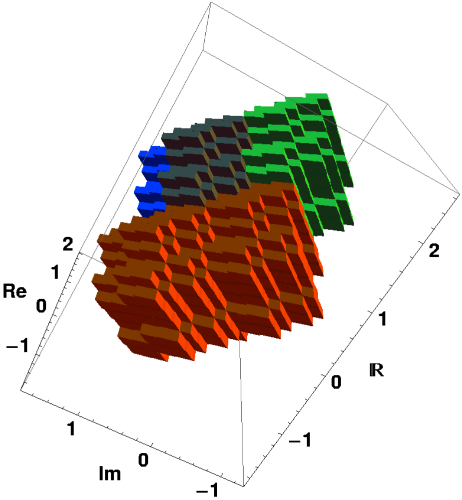

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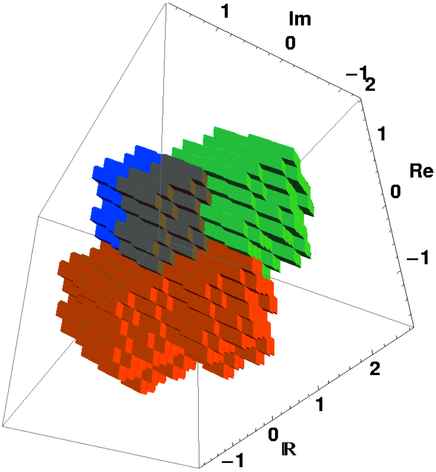

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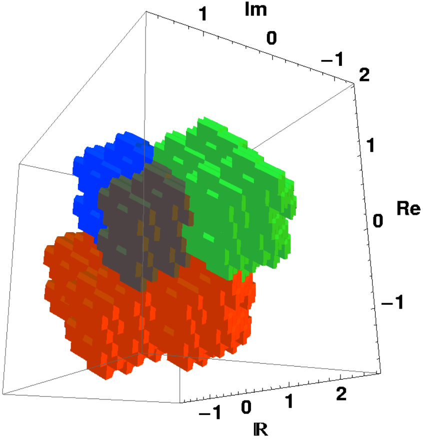

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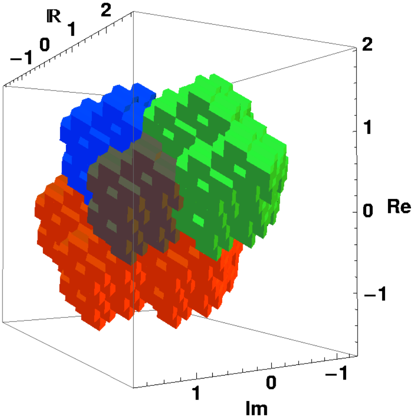

A Non-Unimodular Example

Consider the polynomial $p(x) = x^3 - 3\,x^2 - x - \mathbf{2}$, which has dominant (Pisot!) root $\beta \approx 3.457$.

- Algebraic conjugates are $\lambda_c, \overline{\lambda}_c \approx -0.228 \pm i \cdot 0.726$.
- Associated substitution: $a \longrightarrow aaab$, $b \longrightarrow ac$, $c \longrightarrow aa$.

  Geometrically, we can describe this as a tile substitution:

  Choose $\ell_a = 1$, $\ell_b = \beta - 3$ and $\ell_c = \beta^2 - 3\,\beta - 1$ *etc.*
- However, the measure calculation goes wrong ($\beta\,|\lambda_c|^2 = \mathbf{2}$):

$$\frac{\mathbf{1}}{\mathbf{2}}\,\beta \cdot \begin{pmatrix} \mu\left(\Omega_a\right) \\ \mu\left(\Omega_b\right) \\ \mu\left(\Omega_c\right) \end{pmatrix} \leq \begin{pmatrix} 3 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mu\left(\Omega_a\right) \\ \mu\left(\Omega_b\right) \\ \mu\left(\Omega_c\right) \end{pmatrix}$$

p-adic Fields $\mathbb{Q}_p$

The p-adic integers $\mathbb{Z}_p$ are a complete discrete valuation ring. An element $x \in \mathbb{Z}_p$ can be written as Taylor series in powers of p, i.e.,

$$x = \sum_{n=0}^{\infty} s_n\, p^n = .s_0 s_1 \ldots \qquad \text{with } s_n \in \{0, \ldots, p-1\}.$$

The p-adic numbers $\mathbb{Q}_p$ are the field of fractions of $\mathbb{Z}_p$. An element $x \in \mathbb{Q}_p$ can be written as Laurent series

$$x = \sum_{n=m}^{\infty} s_n\, p^n = s_m \ldots s_{-1}.s_0 s_1 \ldots \qquad \text{with } s_n \in \{0, \ldots, p-1\}.$$

Some 3-adic integers:

$$0 = .\overline{0}, \quad 1 = .1\overline{0}, \quad 8 = .22\overline{0}, \quad -1 = .\overline{2}, \quad -13 = .211\overline{2},$$

$$\frac{3}{4} = .01\overline{20}, \qquad \sqrt{7} = .2112022\ldots \quad \text{and/or} \quad .1110200\ldots.$$

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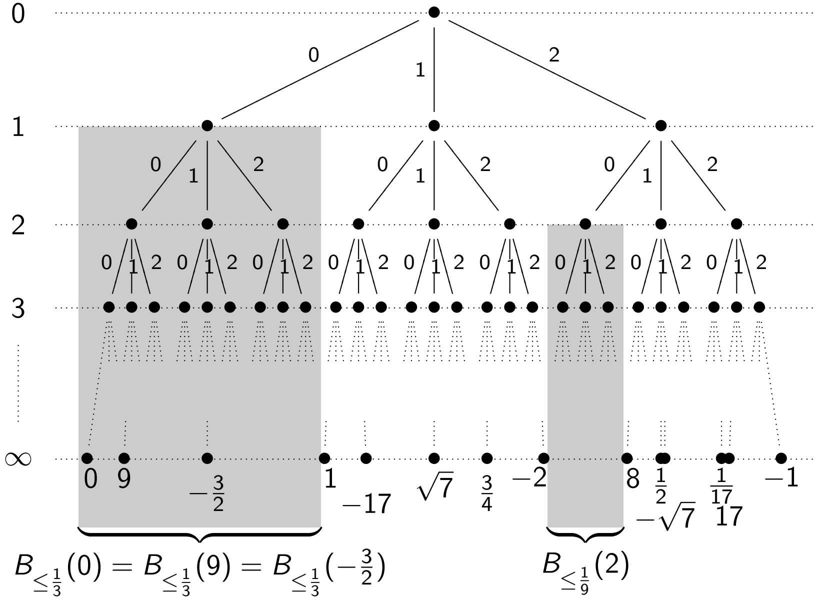
Visualisizing $\mathbb{Q}_3$



Geometric Realization in $\mathbb{C} \times \mathbb{Q}_2$

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We consider the polynomial $p(x) = x^3 - 3\,x^2 - x - 2$ in $\mathbb{Q}_2$:

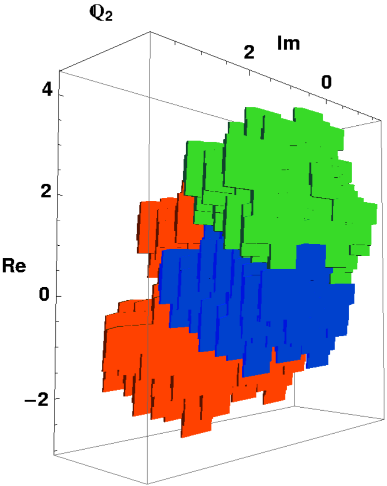
- it has a 2-adic root $\lambda_2 \approx .0101110101011100 0101\ldots$ of absolute value $\frac{1}{2}$!

  (It also has two roots in $\mathbb{Q}_2(\sqrt{-3})$, but they are of absolute value 1.)
- the Haar measure μ_H on $\mathbb{Q}_2$ has the property

$$\mu_H(\lambda_2\, A) = \frac{1}{2} \cdot \mu_H(A).$$

- we here define the $\star$-map $\mathbb{Z}_\beta \to \mathbb{C} \times \mathbb{Q}_2$ by replacing each power β^k by $(\mathrm{Re}(\lambda_c^k), \mathrm{Im}(\lambda_c^k), \lambda_2^k)^T$.
- now the measure calculation (Haar measure on $\mathbb{C} \times \mathbb{Q}_2$) works out!

"Rauzy fractal" $\Omega_a \cup \Omega_b \cup \Omega_c$



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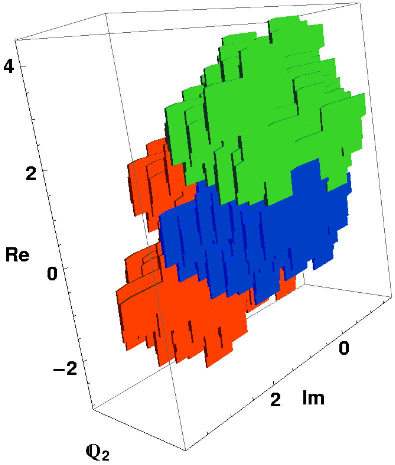

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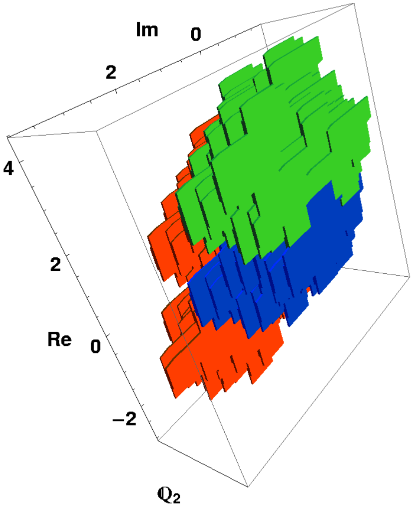

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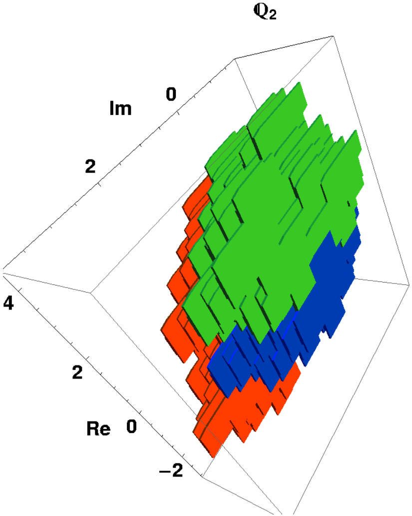

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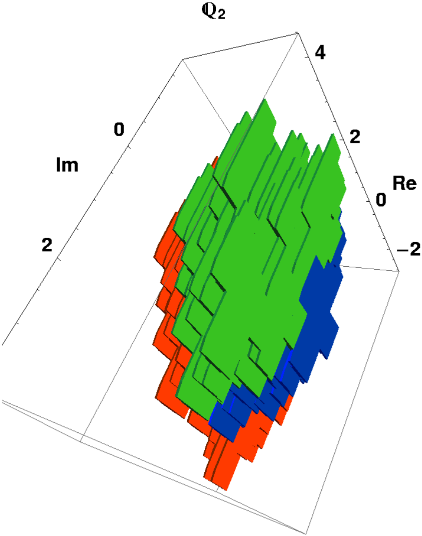

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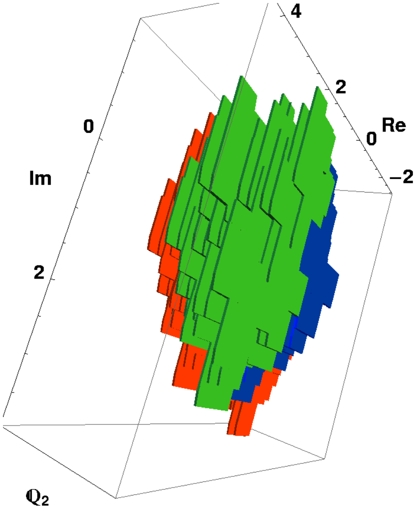

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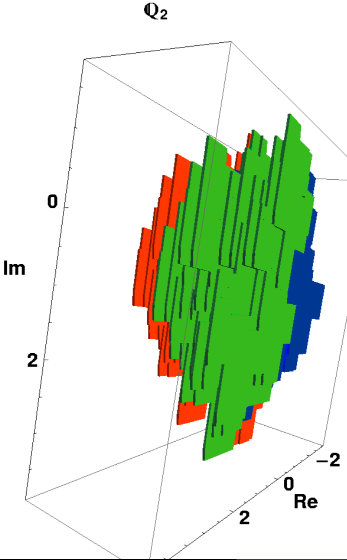

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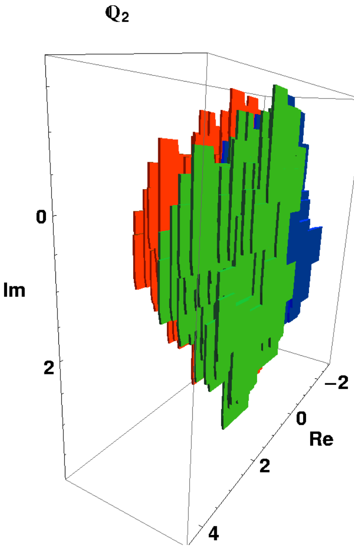

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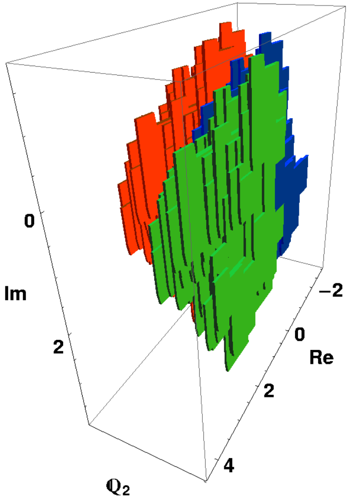

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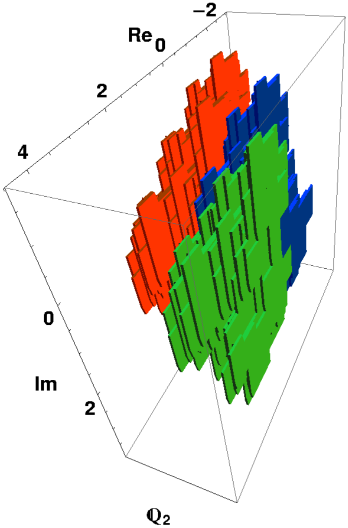

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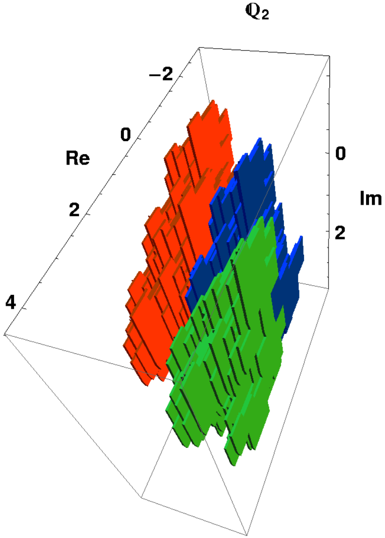

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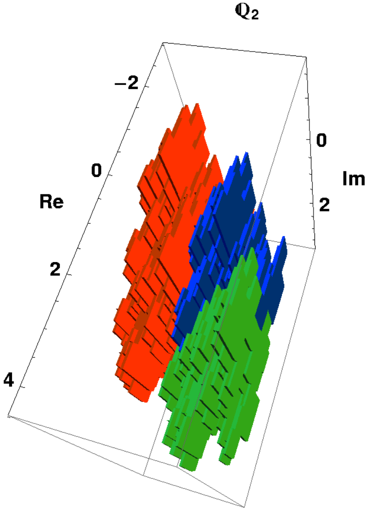

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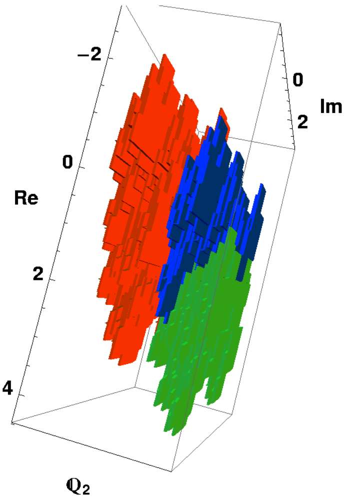

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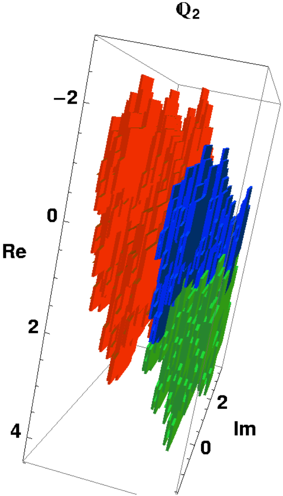

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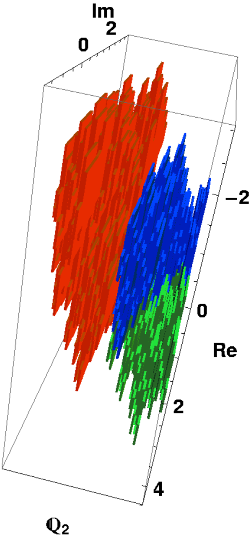

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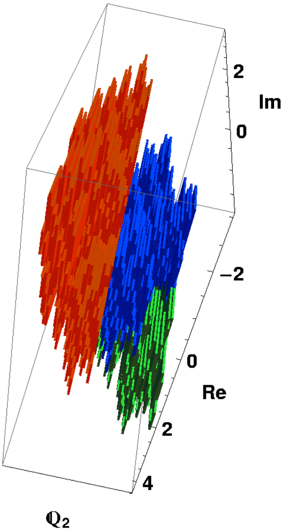

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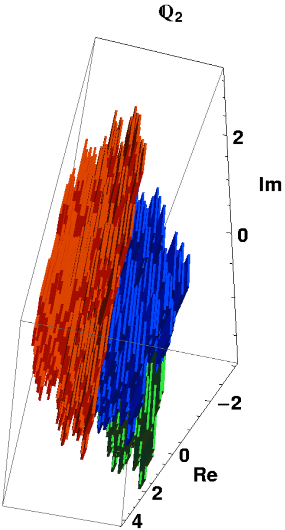

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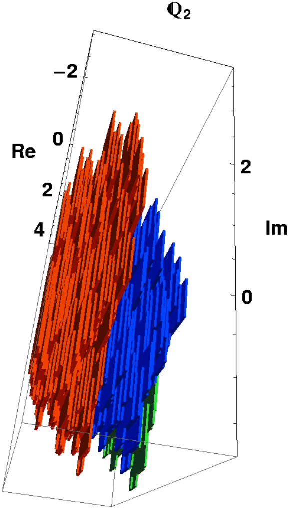

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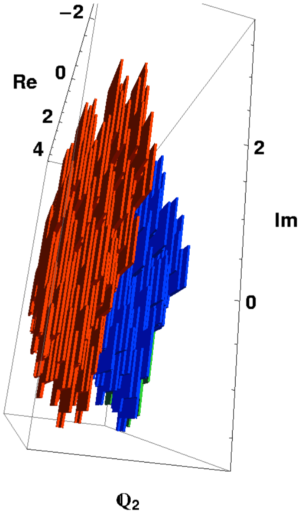
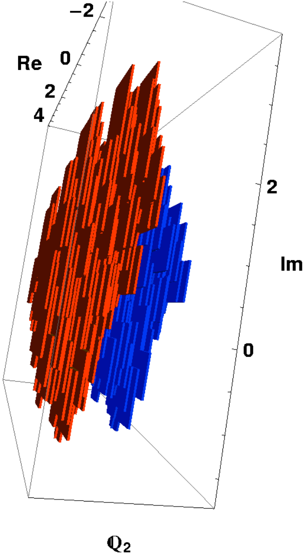


“Rauzy fractal” $\Omega_a \cup \Omega_b \cup \Omega_c$



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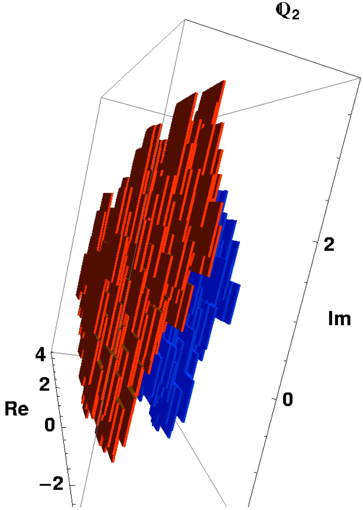

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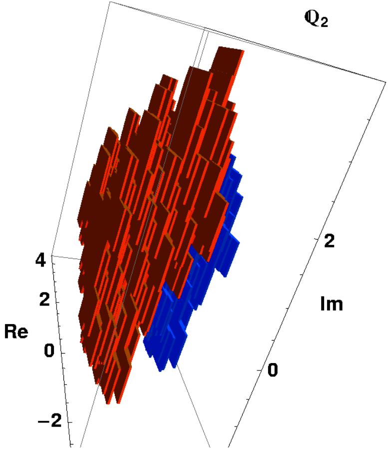

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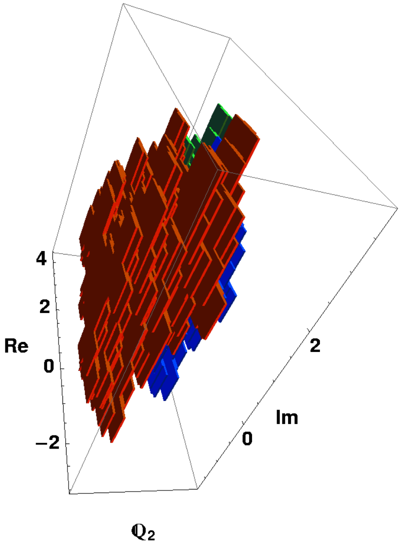

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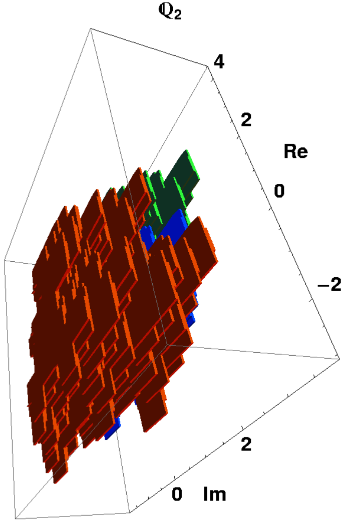

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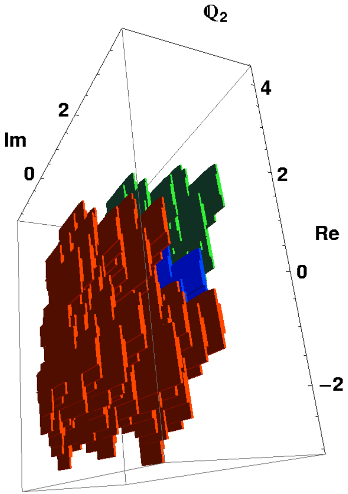

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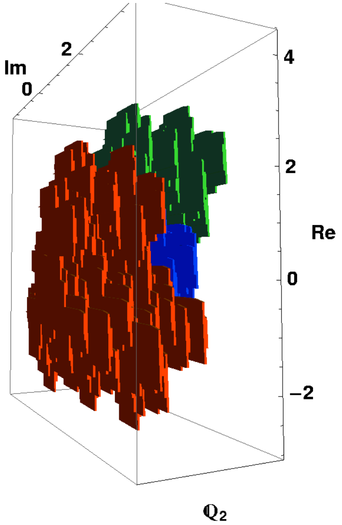

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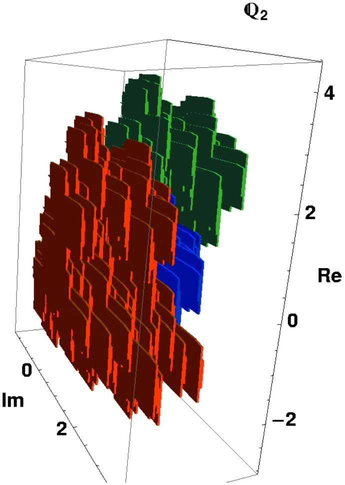

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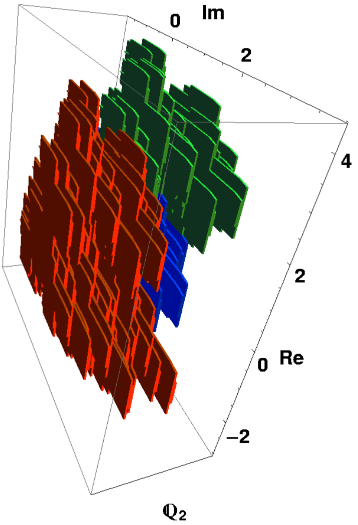

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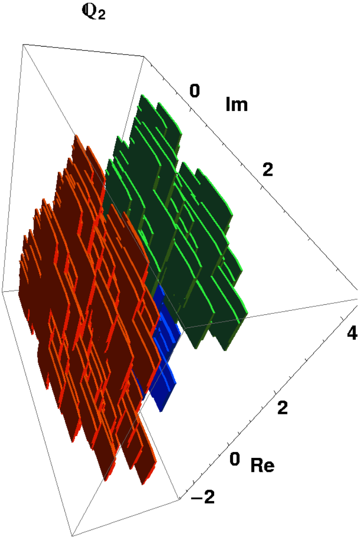

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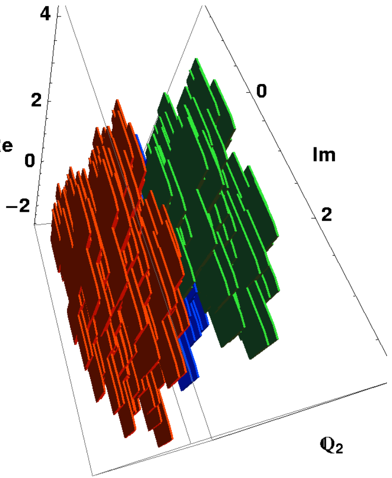

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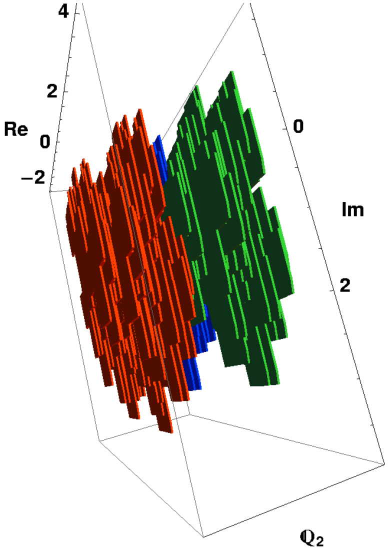

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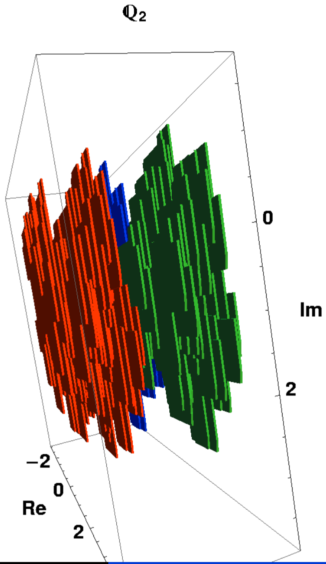

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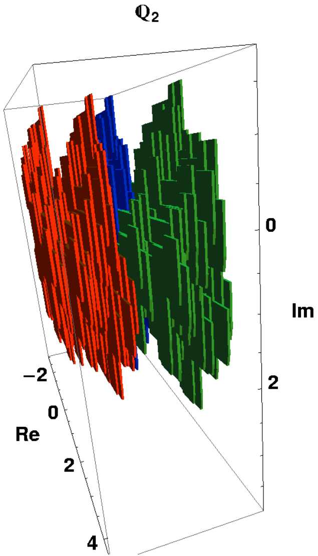

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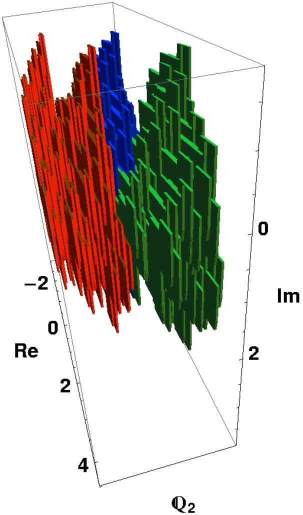

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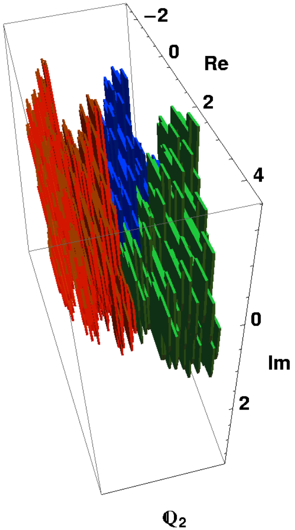

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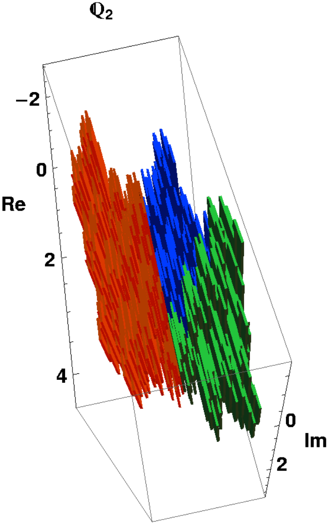

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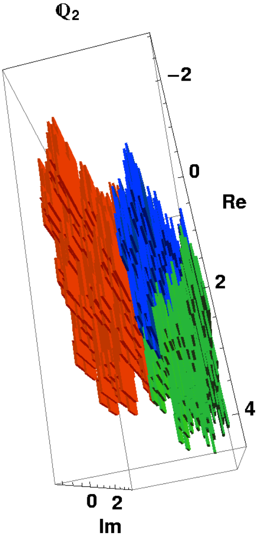

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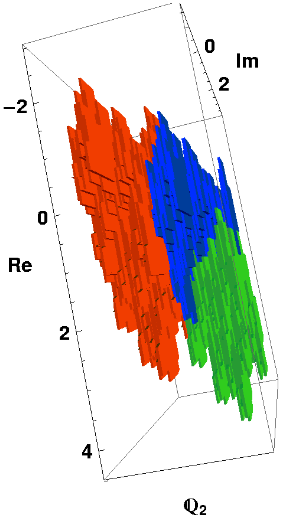

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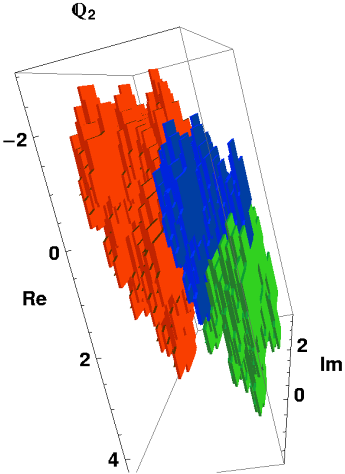

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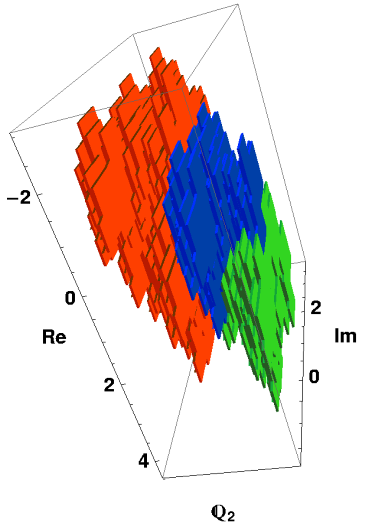

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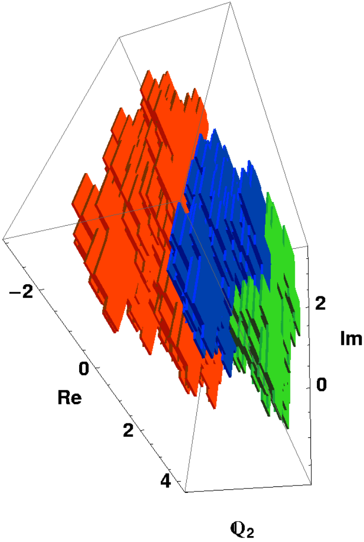

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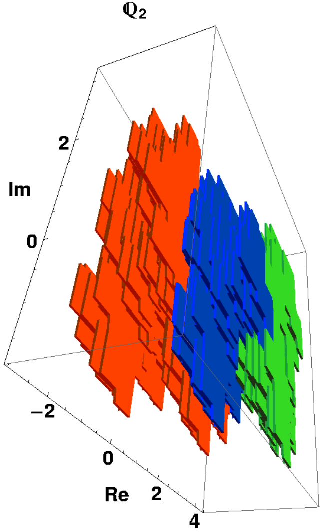

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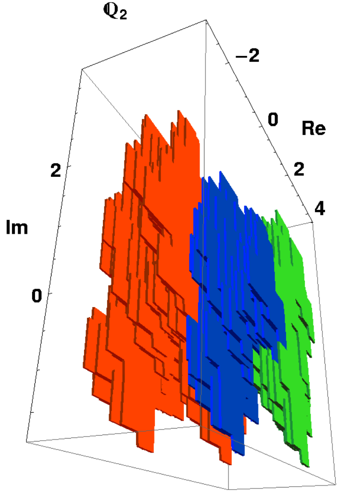

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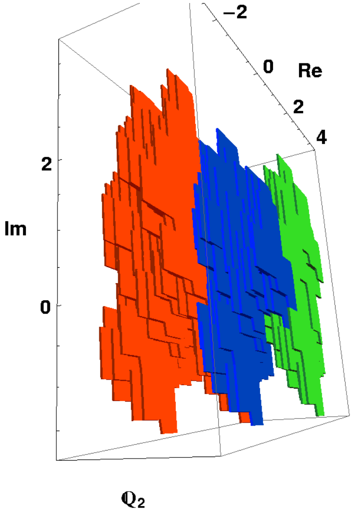

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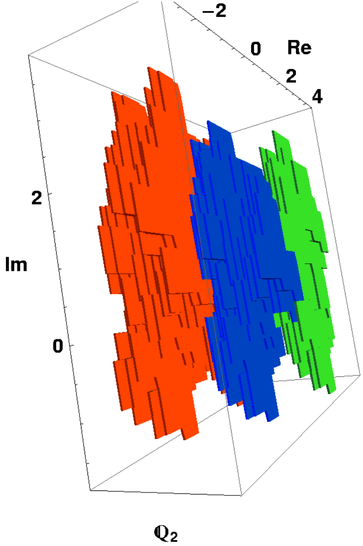

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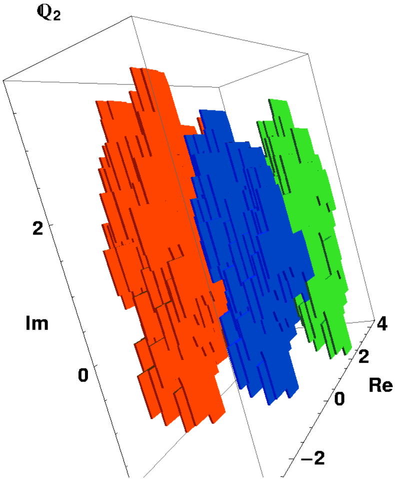

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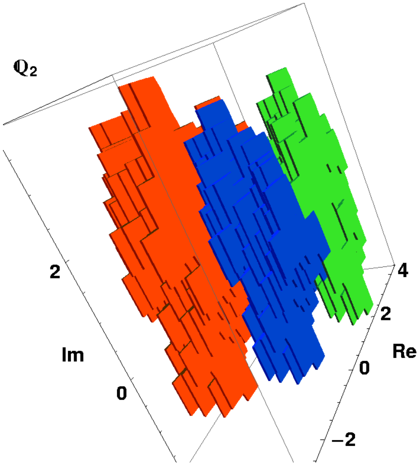

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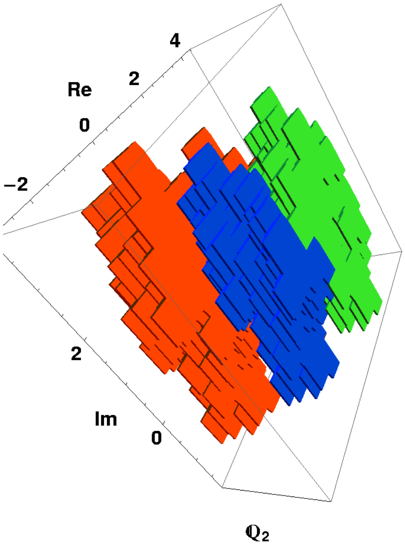

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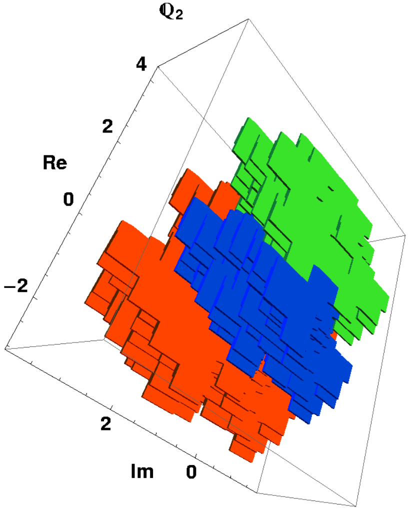

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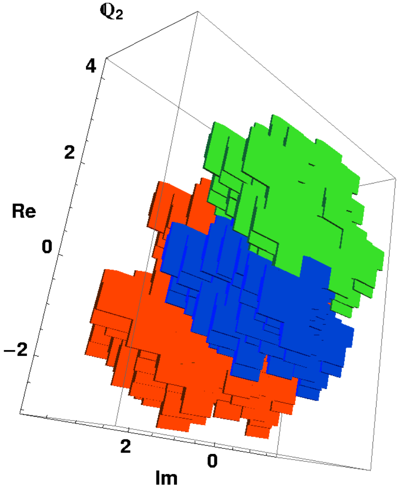

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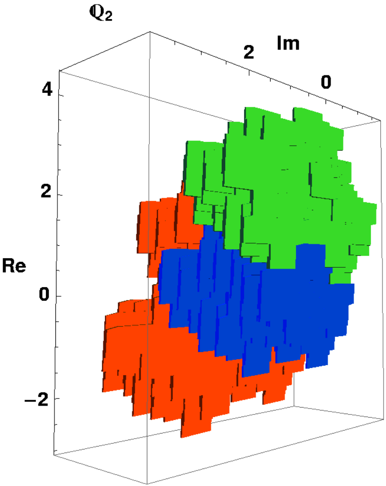


Remark: Coincidences et al.

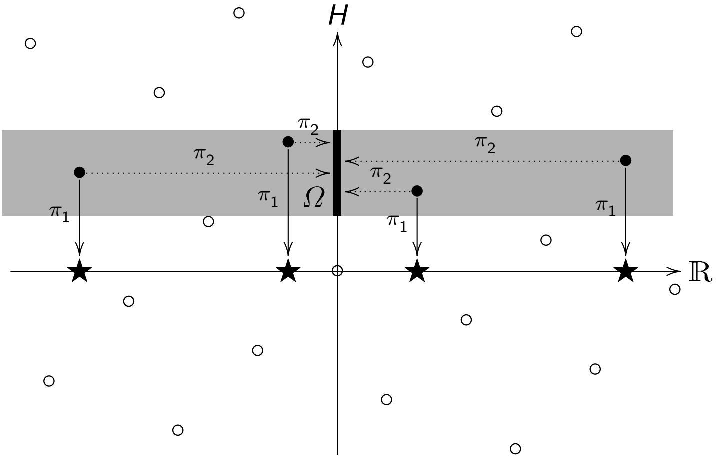
- One is often/usually interested whether the sets Ω_j are the prototiles of a specific aperiodic tiling!

  $\rightsquigarrow$ pure point dynamical system

- One can check this (by an algorithm) by making use of the measure-disjointness on the right-hand side of the iterated function system.

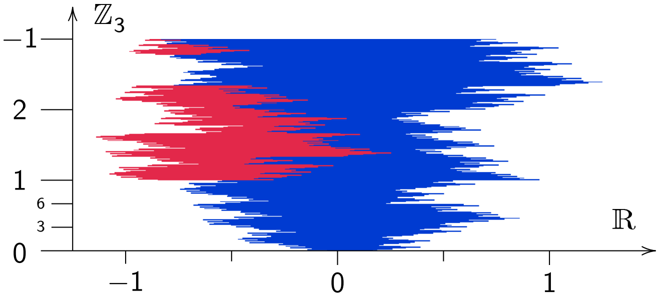
- This yields so-called *coincidence conditions*, *finiteness conditions etc.*

Remark: Cut-and-Project Schemes

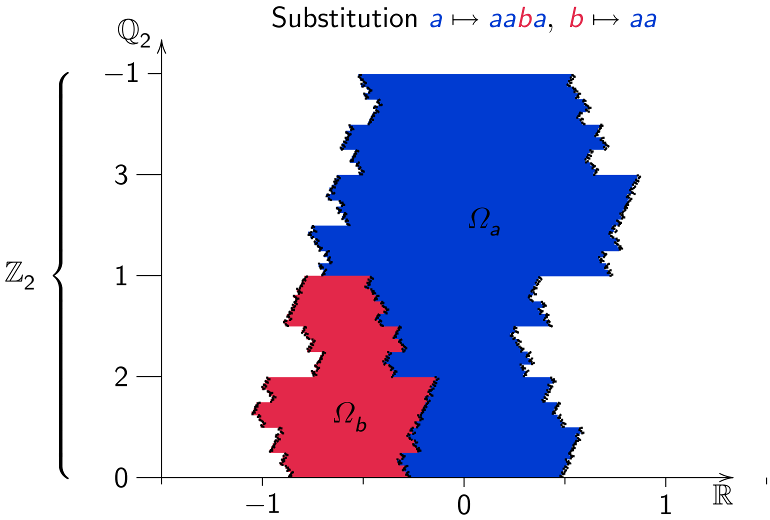


Further Example I

Substitution $a \mapsto aaaab$, $b \mapsto ab$



Further Example II



Further Example II

The aperiodic tiling of $\mathbb{R} \times \mathbb{Q}_2$ with prototiles Ω_a and Ω_b.