Pisot Substitutions and Algebraic Number Theory

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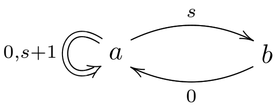
1. Philosophy. For us, Pisot substitutions yield geometrical objects, namely, aperiodic tilings and/or aperiodic point sets, that live in an algebraic number field. Everything else – like dynamical systems, diffraction etc. – comes later.

2. Point and Tile Substitutions. We consider the following examples:

- Silver mean substitution (see Michael Baake's presentation):

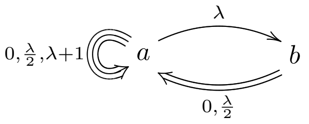
$$\sigma : \begin{array}{l} a \to aba \\ b \to a \end{array}$$



$$M = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{array}{ll} \text{eingenvalues:} & s = 1 + \sqrt{2},\ s' = 1 - \sqrt{2} \\ \text{eigenvectors:} & \begin{pmatrix} s \\ 1 \end{pmatrix}, \begin{pmatrix} s' \\ 1 \end{pmatrix} \end{array}$$

- a non-unimodular example (see [1, Section 6.10.2]):

$$\sigma : \begin{array}{l} a \to aaba \\ b \to aa \end{array}$$



$$M = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} \qquad \begin{array}{ll} \text{eingenvalues:} & \lambda = \frac{3+\sqrt{17}}{2},\ \lambda' = \frac{3-\sqrt{17}}{2} \\ \text{eigenvector:} & \begin{pmatrix} \lambda/2 \\ 1 \end{pmatrix}, \begin{pmatrix} \lambda'/2 \\ 1 \end{pmatrix} \end{array}$$

We recall silver mean here (also see Michael's Baake talk):
The "incoming arrows" in the substitution graph (variant of the prefix graph) tell us how we get left endpoints of the intervals in the tiling:

$$\begin{aligned} \varLambda_a &= s \cdot \varLambda_a \ \cup \ s \cdot \varLambda_a + s + 1 \ \cup \ s \cdot \varLambda_b \\ \varLambda_b &= s \cdot \varLambda_a + s \end{aligned} \tag{1}$$

The "outgoing arrows" tell us what we get from an interval/prototile (multiplying this with the inflation factor s yields the usual tile substitutions):

$$\begin{aligned} A_a &= \tfrac{1}{s} \cdot A_a \ \cup \ \tfrac{1}{s} \cdot (A_a + s + 1) \ \cup \ \tfrac{1}{s} \cdot (A_b + s) \\ A_b &= \tfrac{1}{s} \cdot A_a \end{aligned} \tag{2}$$

We use the *matrix function system* notation (also compare Jeong-Yup Lee's talk) to summarise these two equations as $\underline{\varLambda} = \Theta(\underline{\varLambda})$ and $\underline{A} = \Theta^{\#}(\underline{A})$. Note that all maps in $\Theta^{\#}$ are of the form $x \mapsto \frac{1}{s}(x+t)$ (where $s > 1$ is the inflation factor, t is some translation), and thus contractions with contraction factor $\frac{1}{s}$. But then[1] $\Theta^{\#}$ itself is a contraction (again with factor $\frac{1}{s}$) on $(\mathcal{K}\mathbb{R}, d_H)$, where $\mathcal{K}\mathbb{R}$ denotes the metric space of (nonempty) compact subsets of $\mathbb{R}$ equipped with the Hausdorff

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[1] Also compare [1, Chapter 4] and references therein on iterated function systems.

metric[2]. One calls $\Theta^{\#}$ a *(graph-directed) iterated function system (IFS)* and has by Hutchinson's argument (i.e., applying Banach's Fixed Point Theorem) that there is unique fixed point. This unique fixed "point" is actually a collection of compact sets $\underline{A} = (A_a, A_b)$ and (trivially) given by $A_a = [0, s]$ and $A_b = [0, 1]$.

In fact more is true, and we have the following **Theorem**[3]:
Assume the setting as before; in particular,

- all maps in $\Theta^{\#}$ are of the form $f(x) = \frac{1}{\lambda}x + t$ with some translation t,
- for the Lebesgue measure μ on $\mathbb{R}$ we have $\mu(f(S)) = \frac{1}{|\lambda|} \cdot \mu(S)$ where $S \subset \mathbb{R}$,
- and $\lambda = |\lambda|$ is the PF-eigenvalue of the substitution matrix M.

Then, if one A_k has nonzero measure,

1. all A_i have nonzero measure (by the primitivity of the IFS),
2. unions on the right-hand side of the IFS in Eq. (2) are disjoint in measure (since the above factors of $|\lambda|$ cancel each other),
3. the boundaries ∂A_i have zero measure, the sets A_i are perfect sets and are regularly closed. ☐

For the intervals A_i this is all trivial; however, if we replace s in Eq. (1) by its algebraic conjugate s', we get:

$$\begin{aligned} W_a &= s' \cdot W_a \;\cup\; s' \cdot W_a + s' + 1 \;\cup\; s' \cdot W_b \\ W_b &= s' \cdot W_a + s' \end{aligned} \tag{3}$$

Since $|s'| = \frac{1}{s}$, it is again an IFS and the previous theorem applies[4].

In the non-unimodular example, the intervals $A_a = [0, \frac{\lambda}{2}]$ and $A_b = [0, 1]$ are the solution of a corresponding IFS $\Theta^{\#}$, and replacing λ by its algebraic conjugate λ' in the substitution Θ yields an IFS. However, we have $|\lambda'| = \mathbf{2} \cdot \frac{1}{\lambda}$ – there is an additional factor of 2 (which comes from the minimal polynomial $x^2 - 3x - \mathbf{2}$ for λ) and we cannot apply the stated theorem in the same way as for silver mean.

<u>3. Local Fields.</u> We observe that *everything* (i.e., all numbers in Θ, $\Theta^{\#}$, etc.) "lives" in an algebraic number field $K = \mathbb{Q}(\lambda)$. Recall that the (non-trivial) completions of an algebraic number field K are called *local fields*. Ostrowski's Theorem tells us that a local field is either $\mathbb{R}$ or $\mathbb{C}$ (Archimedean case) or a p-adic field $\mathbb{Q}_p$ or $\mathbb{Q}_{\mathfrak{p}}$ (non-Archimedean/ultrametric case).

The field of p-adic numbers, i.e., the p-adic completion[5] of $\mathbb{Q}$, is given by $\mathbb{Q}_p = \left\{\sum_{n=m}^{\infty} s_n \cdot p^n \mid m \in \mathbb{Z},\ s_n \in \{0, 1, \ldots, p-1\}\right\}$. We write a p-adic number either as $s_m s_{m+1} \ldots s_{-1}.s_0 s_1 s_2 \ldots$ (if $m < 0$) or as $.0 \ldots 0 s_m s_{m+1} \ldots$ (if $m \geq 0$); if $s_m \neq 0$, then the absolute value is given by p^{-m}.

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[2] More precisely, $\Theta^{\#}$ is actually a contraction on the product space $(\mathcal{K}\mathbb{R})^n$.

[3] Exact but somewhat technical formulations can be found in [1, Proposition 4.99 & Corollary 5.63, resp. Corollary 6.66]

[4] The solution of the IFS in Eq. (3) is given by $W_a = \left[\frac{\sqrt{2}-2}{2}, \frac{\sqrt{2}}{2}\right]$ and $W_b = \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}-2}{2}\right]$.

[5] Note that if we consider an algebraic number field $\mathbb{Q}(\lambda)$ (an extension of $\mathbb{Q}$), we might get as its completion some extension of $\mathbb{Q}_p$. Thus, we use the notation $\mathbb{Q}_{\mathfrak{p}}$ where $\mathfrak{p}$ is a prime ideal containing p in the following. For more on local fields, see [1, Chapter 3] and references therein.

We return to the non-unimodular example: 2-adically, the minimal polynomial of λ splits as follows: $x^2 - 3\,x - 2 \;=\; (x - .10110\ldots)\cdot(x - .01101\ldots)$. Thus, we can complete $\mathbb{Q}(\lambda)$ 2-adically (and obtain $\mathbb{Q}_2$ in either case) either by identifying λ with $.10110\ldots$ (with $|.10110\ldots|_2 = 1$) or by identifying λ with $.01101\ldots$ (with $|.01101\ldots|_2 = \frac{1}{2}$).

We now obtain an iterated function system that satisfies our theorem by diagonally embedding the substitution $\underline{\mathit{\Lambda}} = \Theta(\underline{\mathit{\Lambda}})$ into $\mathbb{R}\times\mathbb{Q}_2$ where in the first coordinate λ is interpreted as (the real number) $-0.56\ldots$ (i.e., λ') while in the second coordinate it is (the 2-adic number) $.01101\ldots$. This yields an iterated function system $\underline{W} = \Theta^\star(\underline{W})$ on $\mathcal{K}(\mathbb{R}\times\mathbb{Q}_2)$ where for a set $S\subset\mathbb{R}\times\mathbb{Q}_2$ we have[6] $\mu(f(s)) = |\lambda'|\cdot|.01101\ldots|_2\cdot\mu(S) = \frac{1}{|\lambda|}\cdot\mu(S)$. Then, we can apply our theorem and get "well-behaved" sets W_i.

4. Cut and Project Scheme.

For a general[7] Pisot substitution with Pisot number $\lambda > 1$ of degree n the situation is as follows:

- there are $r-1$ real Galois conjugates of λ that are less than 1 in modulus,
- there are s pairs of complex conjugate Galois conjugates, also less than 1 in modulus (and $n = r + 2s$),
- there are[8] $\mathfrak{p}$-adic fields with $|\lambda|_{\mathfrak{p}} < 1$ only if $p|\det M$ (where p is contained in the prime ideal $\mathfrak{p}$).

We get a *cut and project scheme* with direct space $G = \mathbb{R}$ (the only local field where λ acts as an expansion), internal space $H = \mathbb{R}^{r-1}\times\mathbb{C}^s\times\prod_{\mathfrak{p}:|\lambda|_{\mathfrak{p}}<1}\mathbb{Q}_{\mathfrak{p}}$ (where the diagonal embedding of λ acts as contraction) and a lattice $\tilde{\mathcal{L}}$ in $\mathbb{R}\times H$ that is the diagonal embedding of[9] $\mathcal{L} = \bigcup_{m=0}^{\infty}\frac{1}{\lambda^m}\langle\ell_1,\ldots,\ell_n\rangle_{\mathbb{Z}}$, where ℓ_i denotes the ith interval length and $\langle S\rangle_{\mathbb{Z}}$ the group generated by S.

All this yields a symmetric cut and project scheme $(\mathbb{R}, H, \tilde{\mathcal{L}})$ ($\pi_{\mathbb{R}}, \pi_H$ denote the canonical projections from $\mathbb{R}\times H$ onto $\mathbb{R}$ respectively H):

$$\begin{tikzcd}
\mathbb{R} & \mathbb{R}\times H \arrow[l, "\pi_{\mathbb{R}}"'] \arrow[r, "\pi_H"] & H = \mathbb{R}^{r-1}\times\mathbb{C}^s\times\prod_{\mathfrak{p}:|\lambda|_{\mathfrak{p}}<1}\mathbb{Q}_{\mathfrak{p}} \\
\mathcal{L} \arrow[u, phantom, "\text{dense}\ \cup" description] \arrow[r, leftrightarrow, "1-1"] & \tilde{\mathcal{L}} \arrow[u, phantom, "\cup" description] \arrow[r, leftrightarrow, "1-1"] & \mathcal{L}^\star \arrow[u, phantom, "\cup\ \text{dense}" description]
\end{tikzcd}$$

Thus, the star-map $(\cdot)^\star$ denotes the diagonal embedding into the "contracting" internal space H, and similarly $\tilde{\cdot}$ the diagonal embedding into the product space $\mathbb{R}\times H$.

We also find a symmetric situation by looking at the IFSes (oval boxes) and substitutions (rectangular boxes) on "direct space" $\mathbb{R}$ and "internal space" H:

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[6] Note that the product over all absolute values of a nonzero number x in some number field equals 1.

[7] For this section compare [1, Chapter 6] and references therein.

[8] We only have to consider finitely many p-adic local fields of $\mathbb{Q}(\lambda)$ here: If p does not divide the constant term of the minimal polynomial of λ, which is also given $\pm\det M$, then the corresponding p- respectively $\mathfrak{p}$-adic value is 1, otherwise it is less than or equal to 1.

[9] In the unimodular case, where λ is an algebraic unit, this reduces to $\mathcal{L} = \langle\ell_1,\ldots,\ell_n\rangle_{\mathbb{Z}}$.

On $\mathbb{R}$: $\boxed{\underline{\varLambda} = \Theta(\underline{\varLambda})}$ $\boxed{\underline{A} = \Theta^{\#}(\underline{A})}$ On H: $\boxed{\underline{W} = \Theta^{\star}(\underline{W})}$ $\boxed{\underline{\varUpsilon} = \Theta^{\#\star}(\underline{\varUpsilon})}$

So, an IFS in one space corresponds to a substitution in the other space and vice versa; the unique compact solutions of the IFSes are the intervals A_i and the possible windows/"Rauzy fractals" W_i.

5. Pisot Conjectures. Before stating various equivalent formulations[10] of the *Pisot Substitution Conjecture*, we need to introduce the notion of a *model set*. Let $(G, H, \tilde{\mathcal{L}})$ be a cut and project scheme and assume that $S \subset H$ has nonempty interior and is relatively compact; then the following Delone subset of G is called a model set with window S: $\Lambda(S) = \{\pi_G(x) \mid x \in \tilde{\mathcal{L}}, \pi_H(x) \in S\}$. Since the cut and project scheme above is symmetric, we will write $\Lambda_G(S)$ to emphasise that this model set is a subset of G. If additionally the Haar measure of the boundary ∂S vanishes, we call the model set *regular*.

Pisot Substitution Conjecture I. $\underline{\varLambda}$ is a regular model set with windows $\underline{W}$ (up to boundary points of the sets W_i) – meaning that $\varLambda_i = \Lambda_G(W_i)$ for all i, up to points arising from the boundary of W_i.

By construction, $\underline{\varLambda} + \underline{A} = \{A_i + t \mid t \in \varLambda_i, 1 \le i \le n\}$ is a tiling of $\mathbb{R}$, and some reflection shows that we reformulate our conjecture as follows:

Pisot Substitution Conjecture II. The Pisot Substitution Conjecture holds iff $\Lambda_{\mathbb{R}}(\underline{W}) + \underline{A}$ is a tiling (again, up to boundary points of the sets W_is).

We note that $\Lambda_{\mathbb{R}}(\underline{W}) + \underline{A}$ is always a multi-covering of a.e.-constant covering degree. Using the symmetric structure of the cut and project scheme, we get:

Pisot Substitution Conjecture III. The Pisot Substitution Conjecture holds iff $\Lambda_H(\underline{A}) + \underline{W}$ is a tiling, where $A_i = [0, \ell_i[$ denotes the half-open interval (this makes the model set in H in question repetitive).

The final formulation makes (direct) use of the unique solutions of the IFSes involved:

Pisot Substitution Conjecture IV. The Pisot Substitution Conjecture holds iff $\bigcup_{i=1}^{n} (-A_i) \times W_i$ (and then also $\bigcup_{i=1}^{n} A_i \times (-W_i)$) is a fundamental domain of the lattice $\tilde{\mathcal{L}} \subset \mathbb{R} \times H$.

This, of course, is a nice way to write the torus that plays such an important role for the dynamical systems associated with these tilings. But we stop here.

References

[1] B. Sing, *Pisot Substitutions and Beyond*, PhD-thesis, Universität Bielefeld, 2007. Available at `http://nbn-resolving.de/urn/resolver.pl?urn=urn:nbn:de:hbz:361-11555`.

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[10] These equivalent formulations of the Pisot Substitution Conjecture are amongst those stated in [1, Theorem 6.116]. See there for further references.