MATH1130: Calculus II

Exercise Sheet 2: Scalar and cross product

Please hand solutions in at the lecture on Tuesday 9th February. Attempting Exercise 1 is worth 1% of the final mark.

1*.) (i) Prove Proposition I.2.3 (General Pythagoras): If $\boldsymbol{x}$ and $\boldsymbol{y}$ are perpendicular, then $\|\boldsymbol{x}+\boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2$.

(ii) Prove the the following equation: $\|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2 = 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2$. Also, interpret this equation as "parallelogram law".

2.) Given points $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ in $\mathbb{R}^n$, the vectors $\boldsymbol{y}-\boldsymbol{x}$, $\boldsymbol{z}-\boldsymbol{x}$ and $\boldsymbol{y}-\boldsymbol{z}$ describe the sides of the triangle with vertices at $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$. For each of the following, find the measure of the three angles of the triangle with vertices at the given points.

(i) $\boldsymbol{x} = (1,2,1)$, $\boldsymbol{y} = (-1,-1,2)$ and $\boldsymbol{z} = (-1,3,-1)$.

(ii) $\boldsymbol{x} = (1,2,1,1)$, $\boldsymbol{y} = (-1,-1,2,3)$ and $\boldsymbol{z} = (-1,3,-1,2)$.

3.) (i) Find the area of the parallelogram in $\mathbb{R}^2$ that has vectors $\boldsymbol{x} = (3,1)$ and $\boldsymbol{y} = (1,4)$ for adjacent sides.

(ii) Find the area of the parallelogram in $\mathbb{R}^3$ that has vertices at $(1,1,1)$, $(2,3,2)$, $(-2,4,4)$ and $(-3,2,3)$.

(iii) Find the area of the triangle in $\mathbb{R}^3$ that has vertices at $(1,1,0)$, $(2,3,1)$ and $(-1,3,2)$.

4.) (i) A parallelepiped has base vertices at $(1,1,1)$, $(2,3,2)$, $(-2,4,4)$ and $(-3,2,3)$ and top vertices at $(2,2,6)$, $(3,4,7)$, $(-1,5,9)$ and $(-2,3,8)$. Find its volume.

(ii) Find the volume of the pyramid with vertices $(1,1,-1)$, $(1,2,-2)$, $(2,2,-1)$ and $(1,2,-1)$.

5.) Show by example that the cross product is not associative. That is, find vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ such that

$$\boldsymbol{x} \times (\boldsymbol{y} \times \boldsymbol{z}) \neq (\boldsymbol{x} \times \boldsymbol{y}) \times \boldsymbol{z}.$$

When does $\boldsymbol{x} \times (\boldsymbol{y} \times \boldsymbol{z}) = (\boldsymbol{x} \times \boldsymbol{y}) \times \boldsymbol{z}$ hold?
Hint: For the last question you might find the vector identity $(\boldsymbol{x} \times \boldsymbol{y}) \times \boldsymbol{z} = \langle \boldsymbol{x}, \boldsymbol{z} \rangle \boldsymbol{y} - \langle \boldsymbol{y}, \boldsymbol{z} \rangle \boldsymbol{x}$ useful (you don't have to prove this vector identity!).