MA30056: Complex Analysis

LECTURE NOTES[*]

(UPDATE: APRIL 27, 2009)

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Manifesto

Our Mission

Study functions $f : \mathbb{C} \supset D \to \mathbb{C}$ which are differentiable, i.e., which have, for all $z \in D$, a derivative

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}.$$

This does not look very different from M11, does it?

WRONG!

Consequences

(i) If f is differentiable in the above sense, then it is C^∞! Indeed, it is even C^ω, meaning that it has a power series expansion.

Not so in $\mathbb{R}$: take $f(x) = \int_0^x |\xi| \, \mathrm{d}\xi$; then $f'(x) = |x|$ is not differentiable at $x = 0$.

(ii) A differentiable f (in the above sense) is uniquely determined *inside* a circle by its values *on* the circle.

Completely false in $\mathbb{R}$: a differentiable function is certainly *not* determined by its values on the endpoints of an interval (the "circle" in $\mathbb{R}$).

(iii) If $f : \mathbb{C} \to \mathbb{C}$ is differentiable and $|f|$ is bounded then f is constant.

False in $\mathbb{R}$: consider, for example, $\sin : \mathbb{R} \to \mathbb{R}$.

Applications

(i) An easy proof of the *Fundamental Theorem of Algebra*: Any nonconstant polynomial (with complex coefficients) has a root in $\mathbb{C}$.

(ii) We shall learn quick ways to evaluate tricky series/integrals, like

$$\int_0^\infty \frac{1}{1+x^4} \, \mathrm{d}x = \frac{\pi}{\sqrt{2}} \qquad \text{or} \qquad \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6},$$

and Fourier transforms (important in signal processing, e.g., electrical engineering)

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{1+x^4} \, \mathrm{e}^{-ixt} \, \mathrm{d}x = \frac{e^{-|t|/\sqrt{2}}}{2\sqrt{2}} \left(\cos \frac{|t|}{\sqrt{2}} + \sin \frac{|t|}{\sqrt{2}} \right).$$

(iii) In M9, differential equations are solved using the (one-sided) Laplace transform

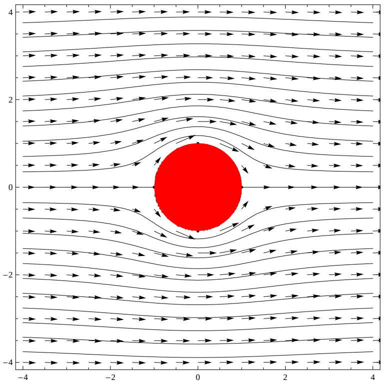
$$\hat{f}(s) = \mathcal{L}\{f\}(s) = \int_0^\infty f(t) \, \mathrm{e}^{-s \cdot t} \, \mathrm{d}t,$$

where $f : [0, \infty) \to \mathbb{R}$ and s is a complex variable. It was noted there that "not all functions possess Laplace transforms; furthermore, the Laplace transform of a function may generally only be defined for certain values of the variable $s \in \mathbb{C}$, e.g., for a function f of exponential order α, the Laplace transform $\hat{f}(s)$ exists for all s with $\mathrm{Re}(s) > \alpha$." These statements are actually statements about the complex differentiability of $\hat{f}(s)$! Moreover, with the knowledge of complex analysis, one does not have to "guess" the inverse Laplace transform (as in M9), but one can use (and make meaning of) the formula

$$f(t) = \mathcal{L}^{-1}\{\hat{f}\}(t) = \frac{1}{2\pi i} \lim_{y \to \infty} \int_{c-iy}^{c+iy} \hat{f}(s)\, \mathrm{e}^{s \cdot t}\, \mathrm{d}s$$

where the path of integration is along the vertical line $\omega \longrightarrow s = c + i\,\omega$ (and c can be any real constant greater than α).

(iv) We can use the theory to study PDE's, like Laplace's equation $\Delta f = 0$ for $f : \mathbb{R}^2 \supset U \to \mathbb{R}$ (useful in physics: see Poisson and heat equation; this can be used to study heat/fluid flow, steady-state temperature distribution etc.).

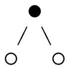


It also lies the foundation stone for many other applications, we mention the following two:

(v) *Combinatorics:* We enumerate binary trees with $n = 0, 1, 2, 3, \ldots$ binary nodes (more precisely, we enumerate ordered rooted binary trees):

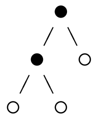
$n = 0:$ ○ $c_0 = 1$

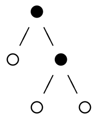
$n = 1:$



$c_1 = 1$

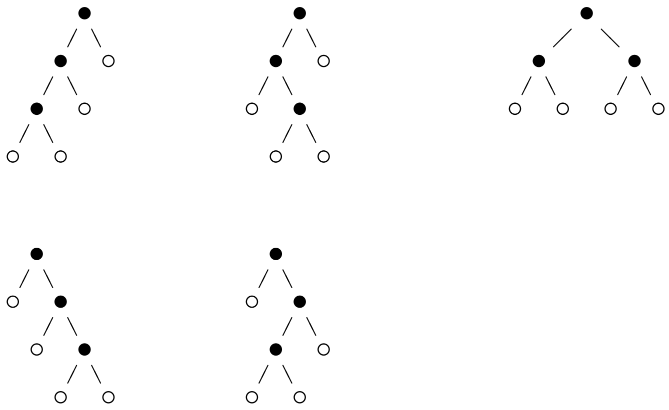
$n = 2:$





$c_2 = 2$

$n = 3:$



$c_3 = 5$

The number c_n (of binary trees with n nodes) is the *nth Catalan number*. One can show that

$$\sum_{n\geq 0} c_n \cdot z^n = \frac{1-\sqrt{1-4z}}{2z}.$$

Thus, a simple series expansion of the function on the right yields the Catalan numbers:

$$\frac{1-\sqrt{1-4z}}{2z} = 1 + z + 2\,z^2 + 5\,z^3 + 14\,z^4 + 42\,z^5 + 132\,z^6 + 429\,z^7 + 1430\,z^8 + \dots$$

Question: How fast does c_n grow as $n \to \infty$?

This can be deduced from the (complex) function $f(z) = \frac{1-\sqrt{1-4z}}{2z}$ using the following "principles of coefficient asymptotics":

- The *location* of a function's *singularities* (i.e., where f is not differentiable) dictates the exponential growth A^n of the coeffictions c_n.
- The *nature* of a function's singularities determines the associated subexponential factor $\Theta(n)$.

The function f in question here, is not differentiable at $z = \frac{1}{4}$ (already as real function we have $\lim_{x\to\frac{1}{4}-} f'(x) = +\infty$) – this yields the exponential growth $\left(\frac{1}{4}\right)^{-n} = 4^n$ – and this singularity is a so-called "square-root singularity" (see the graph of this function), which yields a subexponential factor $\Theta(n) = n^{-3/2}$. Thus, asymptotically one has[1]

$$c_n \sim \frac{1}{\sqrt{\pi}} \cdot 4^n \cdot n^{-3/2} = \frac{4^n}{\sqrt{\pi \cdot n^3}}.$$

And here the graphs of the real function $f(x) = \frac{1-\sqrt{1-4x}}{2x}$ (left) and of $|f(z)|$ over the

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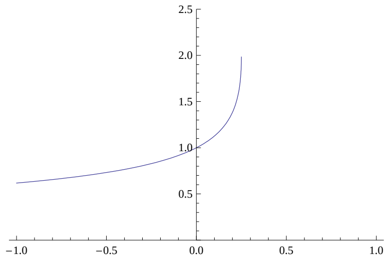
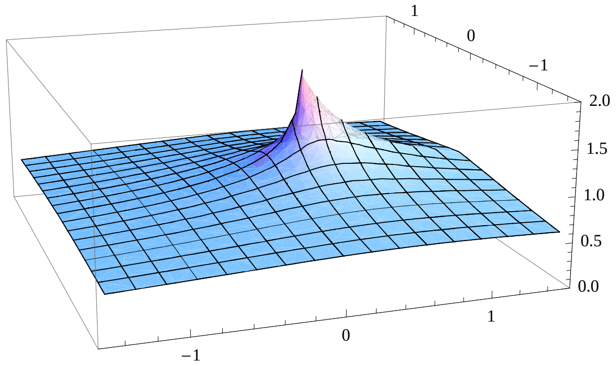
[1] Here, $g(n) \sim h(n)$ means that $\frac{g(n)}{h(n)} \to 1$ as $n \to \infty$.

complex plane (right):





(vi) *Number Theory: Riemann's Zeta Function* $\zeta(s)$ is defined for $\mathrm{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n\in\mathbb{N}} \frac{1}{n^s}.$$

Since any natural number has a unique prime factorization, one easily can establish Euler's product formula

$$\zeta(s) = \sum_{n\in\mathbb{N}} \frac{1}{n^s} = \prod_{p\in\mathbb{P}} \frac{1}{1-p^{-s}},$$

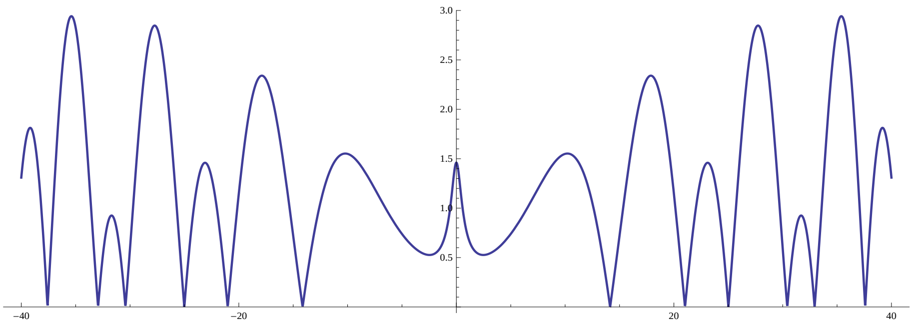
where $\mathbb{P}$ denotes the set of primes. This yields an intriguing connection between Riemann's Zeta Function and the primes.

Now, there is a problem: The above infinite sum (and, similarly, the infinite product) does only converge (absolutely) for $\mathrm{Re}(s) > 1$. However, there exists exactly one complex differentiable function (again called) $\zeta(s)$ defined on $\mathbb{C} \setminus \{1\}$ that coincides with the above function on the half-plane $\mathrm{Re}(s) > 1$.

What does this all have to do with the primes? The statement that $\zeta(s)$ has a singularity (or, more precisely, "a pole of order one") at $s = 1$ implies the statement that there are infinitely many primes. And then there are the zeros of $\zeta(s)$ (or singularities of $1/\zeta(s)$): While there are none in the half-plane $\mathrm{Re}(s) > 1$, there are the so-called "trivial zeros" at $s = -2, -4, -6, -8, -10, \ldots$, while the remaining ("non-trivial" and therefore interesting) zeros lie in the strip $0 \leq \mathrm{Re}(s) \leq 1$. Their location is symmetrical with respect to the *critical line* $\frac{1}{2} + it$. Furthermore, we note:

- Essentially by showing that there are no zeros on the line $1 + it$, Hadamard proved the *Prime Number Theorem* in the year 1896 (de la Vallée Poussin gave independently also a proof in the same year): "The number of primes less than or equal to x is approximately $x/\log(x)$ where the relative error of this approximation approaches 0 as $x \to \infty$."
- The *Riemann Hypothesis* states that all non-trivial zeros of $\zeta(s)$ lie on the critical strip! The Riemann Hypothesis is mentioned in the 8th (of the 23) *Hilbert's problem* and one of the *Clay Mathematics Institute's Millenium Prize Problems* (and thus its proof (not a counterexample!) is worth at least \$1 million).

- The first 10^{13} (and at least 40% of the) non-trivial zeros lie on the critical line – needless to say that no counterexample has been found so far. The correctness of the Riemann Hypothesis, for example, has implications for the error term in the Prime Number Theorem, the distribution of primes would be "quite regular".



Plot of $|\zeta(\frac{1}{2}+it)|$, i.e., of the modulus of Riemann's Zeta Function along the critical line.

I. The Complex Number Plane

I.1. Algebra

References: [DET, Section 1.2] and [ST, Chapter 1]

There are various ways to think about $\mathbb{C}$:

(i) $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$ as a field extension of $\mathbb{R}$, with an “imaginary” unit $i = \sqrt{-1}$ (a solution of $x^2 + 1 = 0$).

(ii) $\mathbb{C} = \{(x, y) \in \mathbb{R}^2\}$ as vector space with multiplication

$$(x, y) \cdot (u, v) = (xu - yv, xv + yu).$$

(iii) $\mathbb{C} = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\} \subset M(2 \times 2, \mathbb{R})$ with the usual addition and multiplication of matrices.

These models are all *isomorphic*, i.e., the obvious maps between them are bijections that preserve the algebraic operations. However, each has something to tell:

(i) provides clear and simple (historical) notation,

(ii) provides an important geometric interpretation,

(iii) provides a simple argument that $\mathbb{C}$ is a field.

We shall use the notation of (i) and the geometric interpretation of (ii) (the *Gauss' complex number plane*).

(ex) Verify that the three models of $\mathbb{C}$ are isomorphic.

(ex) Verify that $\mathbb{C} = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$ is (with the usual addition and multiplication) a field.

Complex conjugation. If $z = x + iy \in \mathbb{C}$ its *complex conjugate* is

$$\overline{z} = \overline{x + iy} = x - iy.$$

Note that $\mathbb{C} \ni z \mapsto \overline{z} \in \mathbb{C}$ satisfies (ex)

$$\overline{z + w} = \overline{z} + \overline{w} \quad \text{and} \quad \overline{z \cdot w} = \overline{z} \cdot \overline{w}.$$

Real and imaginary parts. If $z = x + iy \in \mathbb{C}$ these are defined as

$$\operatorname{Re} z = x \quad \text{and} \quad \operatorname{Im} z = y.$$

We say that z is *real* if $\operatorname{Im} z = 0$ and z is *imaginary* if $\operatorname{Re} z = 0$.
ex Verify that $\operatorname{Re} z = \frac{1}{2}(z + \overline{z})$ and $\operatorname{Im} z = \frac{1}{2i}(z - \overline{z})$.

Modulus. The *modulus* of $z = x + iy \in \mathbb{C}$ is its Euclidean length

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\overline{z}}.$$

Note that, if $z \neq 0$, then $1 = z \cdot \frac{\overline{z}}{|z|^2}$, that is, z has an *inverse* $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.

ex Let $z, w \in \mathbb{C}$. Verify that

(i) $|\operatorname{Re} z|, |\operatorname{Im} z| \leq |z|$,
(ii) $|z| = |\overline{z}|$, i.e., $\mathbb{C} \ni z \mapsto \overline{z} \in \mathbb{C}$ is an *isometry*,
(iii) $|z \cdot w| = |z| \cdot |w|$, i.e., $|\,.\,| : (\mathbb{C}, \cdot) \to (\mathbb{R}, \cdot)$ is a *homomorphism*,
(iv) $|z| \geq 0$ and $|z| = 0$ iff $z = 0$.

Conclude that the *triangle inequality* $|z + w| \leq |z| + |w|$ holds.
Hint: compute $|z + w|^2$.

ex Prove that $|\,.\,| : \mathbb{C} \to \mathbb{R}$ is continuous.

ex Show that $\mathbb{C}$ is *not*(!) an *ordered field*.
Note that an *ordering* of a field K is a subset $P \subset K$ having the following properties:

(O1) Given $x \in K$, we have either $x \in P$, or $x = 0$ or $-x \in P$, and these three possibilities are mutually exclusive. In other words, K is the disjoint union of P, $\{0\}$ and $-P$.
(O2) If $x, y \in P$, then $x + y \in P$ and $x \cdot y \in P$.

We shall also say that K is *ordered by* P, and we call P the set of *positive elements*.
Hint: Proof by contradiction.

I.2. Geometry

References: [DET, Section 1.3] and [ST, Chapter 1]

When we think of $\mathbb{C} \cong \mathbb{R}^2$ with a multiplication then:

(i) addition is vector addition,
(ii) conjugation is reflection in the real axis $\operatorname{Im} z = 0$,
(iii) the modulus is the Euclidean length of the vector z,
(iv) multiplication of z by a *real* number w, $\operatorname{Im} w = 0$, is scalar multiplication.

Question: How to think about multiplication by a complex number?

Polar form of a complex number. This is given by

$$z = r\,(\cos\vartheta + i\sin\vartheta),$$

where (r, ϑ) are the polar coordinates of $z = x + iy = (x, y)$.
(ex) Write $z, w \in \mathbb{C}$ in polar form

$$z = r\,(\cos\vartheta + i\sin\vartheta) \quad \text{and} \quad w = s\,(\cos\varphi + i\sin\varphi);$$

then

$$z \cdot w = rs\,(\cos(\vartheta + \varphi) + i\sin(\vartheta + \varphi)).$$

Answer (to the above question):

(iv)' Multiplication of z by $w = s\,(\cos\varphi + i\sin\varphi) \in \mathbb{C}$ is a stretch-rotation:

- stretching by $s = |w|$ followed by a
- counter-clockwise rotation by the angle φ.

(ex) Show that the equation $z^n = 1$, $n \in \mathbb{N}$, has n solutions. Determine the solutions of $z^3 = 1$.

I.3. Topology

References: [DET, Section 1.4] and [ST, Section 2.1]
(Bolzano-Weierstrass: [DET, Theorem 1.4.1]; Heine-Borel: [DET, Theorem 1.4.3])

Overall theme: $\mathbb{C}$ is homeomorphic (even isometric) to $\mathbb{R}^2$ (both have the “same” topology (i.e., open/closed sets), (Cauchy/convergent) sequences etc.).

Notation:

- open disk/ball $B_r(z) = \{w \in \mathbb{C} \mid |w - z| < r\}$,
- closed disk/ball $\overline{B}_r(z) = \{w \in \mathbb{C} \mid |w - z| \leq r\}$.

If $z = 0$, it is sometimes dropped, i.e., we write B_r and $\overline{B}_r$ instead of $B_r(0)$ respectively $\overline{B}_r(0)$

Recall. A subset $A \subset \mathbb{C}$ ($\cong \mathbb{R}^2$) is called

- *open* if $\forall z \in A\ \exists \varepsilon > 0 : B_\varepsilon(z) \subset A$ (or if $A = \varnothing$),
- *closed* if $\mathbb{C} \setminus A$ is open.

Definition. A subset $M \subset A \subset \mathbb{C}$ is called

- *open in* A if $\forall z \in M\ \exists \varepsilon > 0 : B_\varepsilon(z) \cap A \subset M$,
- *closed in* A if $A \setminus M \subset A$ is open in A.

(ex) Show that open/closed discs are open/closed.

Convergence. $(z_n)_{n\in\mathbb{N}}$ *converges* to $z \in \mathbb{C}$ if $|z_n - z| \to 0$ as $n \to \infty$.

ⓔⓧ Show that $z_n \to z$ iff $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall n \geq N : z_n \in B_\varepsilon(z)$.

ⓔⓧ Prove that $M \subset A \subset \mathbb{C}$ is closed in A iff it contains the limit of every sequence $(z_n)_{n\in\mathbb{N}} \subset M$ that converges in A.

Remark. If $A = \mathbb{C}$ the previous exercise yields the usual sequential characterization of closed subsets $M \subset \mathbb{C}$.

Lemma I.3.1. *If* $z_n = x_n + iy_n$ *and* $z = x + iy$ *then*

$$z_n \to z \qquad \Leftrightarrow \qquad x_n \to x \quad \text{and} \quad y_n \to y.$$

ⓔⓧ Prove Lemma I.3.1
Hint: Use $|x_n - x|, |y_n - y| \leq |z_n - z|$.

Now, the main theorems from M7/M11 carry over.

Recall. A sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ $(\cong \mathbb{R}^2)$ is called

- *Cauchy* if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N : |z_n - z_m| < \varepsilon$;
- *bounded* if $\exists R \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N} : z_n \in B_R(0)$.

Theorem (Completeness of $\mathbb{C}$)**.** *A sequence* $(z_n)_{n\in\mathbb{N}}$ *is convergent iff it is a Cauchy sequence.*

ⓔⓧ Prove the Completeness Theorem.
Hint: "⇒" (every convergent sequence is Cauchy): usual $\frac{\varepsilon}{2}$-argument.
"⇐" (completeness): use Lemma I.3.1.

Theorem (Bolzano-Weierstrass)**.** *Any bounded sequence in* $\mathbb{C}$ *has a convergent subsequence.*

ⓔⓧ Prove the Bolzano-Weierstrass Theorem.
Hint: Use Bolzano-Weierstrass in $\mathbb{R}$ twice, first for (x_{n_i}), then for $(y_{n_{i_j}})$, and then use Lemma I.3.1.

Recall. A subset $K \subset \mathbb{C}$ is compact if every sequence in K has a convergent subsequence with limit in K.

Theorem ((Sequential) compactness in $\mathbb{C}$)**.** $K \subset \mathbb{C}$ *is* compact *iff every covering of* K *by open sets has a finite subcovering.* □

Remark. This is sometimes taken as definition of compactness.

In $\mathbb{C}$ (and $\mathbb{R}^n$, $\mathbb{C}^n$) we have an easier characterisation of compactness.

Theorem (Heine-Borel in $\mathbb{C}$)**.** $K \subset \mathbb{C}$ *is* compact *iff it is closed and bounded.*

Proof. "⇒": use sequential characterizations.
"⇐": Use Bolzano-Weierstrass. □

ⓔⓧ (Cantor's Intersection Theorem) Let $K_n \subset \mathbb{C}$ be compact with $K_n \supset K_{n+1}$ for all n and

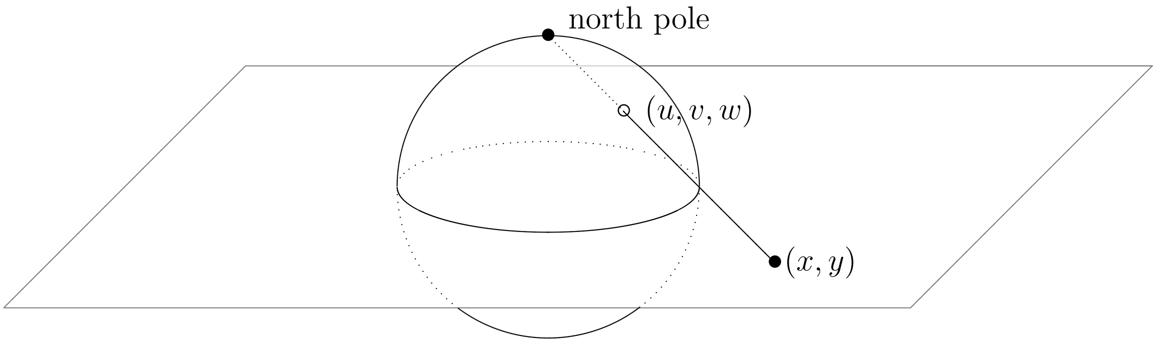
$$\operatorname{diam} K_n = \sup_{z,w\in K_n} |z - w| \to 0 \quad \text{as} \quad n \to \infty.$$

Prove that $\exists z \in \mathbb{C} : \bigcap_{n\in\mathbb{N}} K_n = \{z\}$.

I.4. Stereographic Projection (Not examinable!)

Reference: [DET, Section 1.5] and [ST, Section 11.4]

A complex number $z = x + iy \in \mathbb{C}$ can be represented as point (x, y) in the plane $\mathbb{R}^2$. One can also associate a point (u, v, w) on the unit sphere $\mathbb{S} = \{(u, v, w) \in \mathbb{R}^3 \,|\, u^2+v^2+w^2 = 1\}$, called the *Riemann sphere* in this context, with a given point (x, y) in the plane[1]. The associated mapping is called *stereographic projection*.



We note:

- A point (u, v, w) on the sphere corresponds to (x, y) if the north pole $(0, 0, 1)$, (u, v, w) and $(x, y, 0)$ are on a line.
- The equator of the sphere corresponds to the unit circle in the plane.
- The south pole $(0, 0, -1)$ corresponds to the origin $(0, 0)$.

Definition. The *stereographic projection*, which projects a point $(u, v, w) \in \mathbb{S} \setminus \{0, 0, 1\}$ to a point of the (complex) plane $z = x + iy \in \mathbb{C} \cong \mathbb{R}^2$, and its inverse are given by the following maps:

$$u = \frac{2\,x}{|z|^2 + 1}, \qquad v = \frac{2\,y}{|z|^2 + 1}, \qquad w = \frac{|z|^2 - 1}{|z|^2 + 1}$$

and

$$x = \frac{u}{1 - w}, \qquad y = \frac{v}{1 - w}.$$

Remark. Under stereographic projection we have:

$\mathbb{C}$ $\overset{\text{continuous}}{\longleftrightarrow}$	$\mathbb{S} \setminus \{(0, 0, 1)\}$
with the topology of $\mathbb{R}^2$	with the topology of $\mathbb{R}^3$, i.e., metric is given by the Euclidean metric in $\mathbb{R}^3$ (*chordal metric* on $\mathbb{S}$)
is complete	not complete (closure $\mathbb{S}$ is complete) (there are sequences converging to the north pole)
closed and open	open in $\mathbb{S}$, but not closed
not compact	closure $\mathbb{S}$ is compact

Obviously, the north pole plays a very special role!

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[1] We embed the plane into $\mathbb{R}^3$ by $(x, y) \mapsto (x, y, 0)$.

Definition. $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called the *extended complex plane*, where ∞ denotes the *point at infinity* (its image point on the Riemann sphere is the north pole).

Remark.

- A sequence $(z_n)_{n\in\mathbb{N}}$ is unbounded in $\mathbb{C}$ iff there exists a subsequence $(z_{n_k})_{k\in\mathbb{N}}$ s.t. $z_{n_k} \to \infty$ in $\widehat{\mathbb{C}}$.
- A straight line in $\mathbb{C}$ corresponds to a circle through ∞ in $\hat{\mathbb{C}}$.
- In topology, $\widehat{\mathbb{C}}$ is called the *one-point compactification* of $\mathbb{C}$.

I.5. Curves and Regions

References: [DET, Section 1.6] and [ST, Sections 2.4–2.6]
(Jordan Curve Theorem: [DET, Theorem 1.6.1])

Definition. A *path* is a continuous map $\gamma : [a, b] \to \mathbb{C}$. It is called

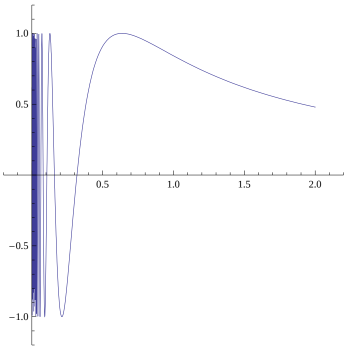
- *simple* if γ is injective,
- *closed* if $\gamma(a) = \gamma(b)$, and
- *simple closed* if $\gamma(s) = \gamma(t) \;\Leftrightarrow\; s = t$ or $\{s, t\} = \{a, b\}$.

$\Gamma \subset \mathbb{C}$ is called a *(simple/closed) Jordan curve* if $\Gamma = \gamma([a, b])$ for some (simple/closed) path $\gamma : [a, b] \to \mathbb{C}$.

Definition. $A \subset \mathbb{C}$ is called *path connected* if any two points $z, w \in A$ can be joined by a path $\gamma : [a, b] \to A$ in A.

Remark. Path connectedness implies connectedness in the topological sense ($A \subset \mathbb{C}$ is connected if it is not the union of two disjoint subsets that are open in A, i.e., there are no nonempty sets A_1, A_2 open in A s.t. $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \varnothing$). The converse is not true: take $\textcircled{ex}$

$$A = \{z = x+iy \in \mathbb{C} \,|\, (x = 0 \text{ and } y \in [-1, 1]) \vee (y = \sin \tfrac{1}{x})\}.$$



Theorem I.5.1. *Let $A \subset \mathbb{C}$ be open. Then A is connected iff it is path connected (iff it is* polygonally/step connected[2]*).*

Proof. Let A be connected and let $z_0 \in A$ and define $A_1 = \{z \in A | \exists$ a path joining z and $z_0\}$ and $A_2 = A \setminus A_1$ (A_1 is the *connected component* of the point z_0). Then, one can show that A_1, A_2 are open $\textcircled{ex}$, therefore they are open and closed in A. They are also disjoint and their union is all of A. So, since A is connected and A_1 is nonempty, A_2 must be the empty set. Thus, $A = A_1$ and A is path connected. □

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[2] A set A is *polygonally connected* if for any two points $z, z' \in A$ there is a polygonal line joining them which is contained in A. If, furthermore, each straight line segment in this polygonal line is parallel to either the real or the imaginary axis, then the set is called *step connected*, see [ST, Section 2.6].

The last part in the previous proof is actually an alternative characterisation of connectedness: Let $A \subset \mathbb{C}$ be connected and $A_1 \subset A$ open and closed in A. Then $A_1 = A$ or $A_1 = \varnothing$. A similar statement holds if we replace "connected" by "path connected".

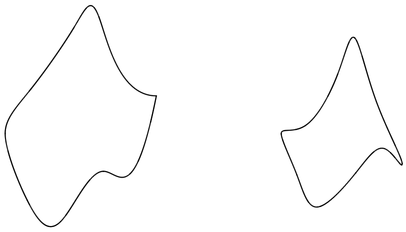
Lemma I.5.2. *Let $A \subset \mathbb{C}$ be path connected and $A_1 \subset A$ open and closed in A. Then $A_1 = A$ or $A_1 = \varnothing$.*

ex Prove Lemma I.5.2.
Hint: Proof by contradiction.

Definition. A nonempty (path) connected open subset $D \subset \mathbb{C}$ is a *domain*[3].

We now look at simple closed Jordan curves in $\mathbb{C}$.

How can we discriminate the "inside" from the "outside"?

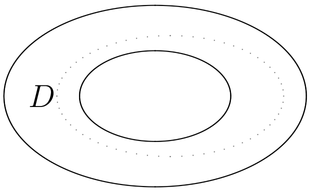


Theorem (Jordan Curve Theorem). *If $\Gamma \subset \mathbb{C}$ is a simple closed Jordan curve then $\mathbb{C} \setminus \Gamma$ is the disjoint union of exactly two domains, one of which (the "interior") is bounded and the other (the "exterior") is unbounded.* □

Remark. This is a deep theorem which we cannot prove in this course.

The next property means that D has "no holes".

Definition. A domain $D \subset \mathbb{C}$ will be called *simply connected*[4] if the interior I_Γ of every simple closed Jordan curve $\Gamma \subset D$ lies inside D, i.e., $I_\Gamma \subset D$.



not simply connected

Restriction of paths. If $\gamma : [a, b] \to \mathbb{C}$ is a path and $[c, d] \subset [a, b]$ then $\gamma|_{[c,d]} : [c, d] \to \mathbb{C}$ is clearly a path.

Composition of paths. If $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ are paths with $\gamma_1(b) = \gamma_2(c)$, then

$$[a, b+d-c] \ni t \mapsto \gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [a, b] \\ \gamma_2(t-b+c) & \text{if } t \in [b, b+d-c] \end{cases}$$

is the path $\gamma = \gamma_1 + \gamma_2$.

---

[3] A *region* is a domain plus none/any/all of its boundary points.

[4] If you know what "homotopic" means: D is simply connected iff every closed path in D is *homotopic* to a "null path"/point curve (i.e., a single point) in D.

Inverse of a path. If $\gamma : [a,b] \to \mathbb{C}$ is a path then the "reversed"/opposite path $-\gamma$ is

$$[a,b] \ni t \mapsto (-\gamma)(t) = \gamma(a+b-t).$$

Definition. A path $\gamma : [a,b] \to \mathbb{C}$ is called

- *smooth* if $\gamma'(t) = \lim_{s\to t, s\in[a,b]} \frac{\gamma(s)-\gamma(t)}{s-t}$ exists for all $t \in [a,b]$ and is continuous on $[a,b]$;
- *regular* if it is smooth and $\gamma'(t) \neq 0$ for all $t \in [a,b]$;
- *piecewise smooth (regular)* if it is the composition of finitely many smooth (regular) paths.

$\mathit{\Gamma} \subset \mathbb{C}$ will be called *(simple/closed) contour* if $\mathit{\Gamma} = \gamma([a,b])$ with some (simple/closed) piecewise regular path; γ is a *parametrization* of $\mathit{\Gamma}$.

(ex) Prove that the composition of two (piecewise smooth) paths is a (piecewise smooth) path.

Example. A polygon is a contour. If it is closed and has no self-intersections, then it is a simple closed contour.

Reparametrizations of contours. Let $\gamma : [a,b] \to \mathbb{C}$ be a regular path, and $\mathit{\Gamma} = \gamma([a,b])$.

If $h : [c,d] \to [a,b]$ is onto and regular, then $\tilde{\gamma} = \gamma \circ h : [c,d] \to \mathit{\Gamma} \subset \mathbb{C}$ is a new regular parametrization of $\mathit{\Gamma}$.

If $\tilde{\gamma} : [c,d] \to \mathit{\Gamma} \subset \mathbb{C}$ is another regular parametrization of $\mathit{\Gamma}$ and $\mathit{\Gamma}$ is simple then there is a regular $h : [\tilde{a}, \tilde{b}] \to [a,b]$ so that $\tilde{\gamma} = \gamma \circ h$.

Note: the second statement is not very easy to prove (how to make $\gamma^{-1} : \mathit{\Gamma} \to [a,b]$ differentiable?); a possible proof uses the Inverse Function Theorem.

Length of a contour. If $\gamma : [a,b] \to \mathbb{C}$ is a regular path then the length of $\mathit{\Gamma} = \gamma([a,b])$ is given by $\int_{\mathit{\Gamma}} \mathrm{d}s = \int_a^b |\gamma'(t)| \, \mathrm{d}t$.

The length of a contour is the sum of the lengths of its regular pieces.

II. Functions of a Complex Variable

II.1. Functions and Limits

References: [DET, Section 2.1][1] and [ST, Section 2.3]

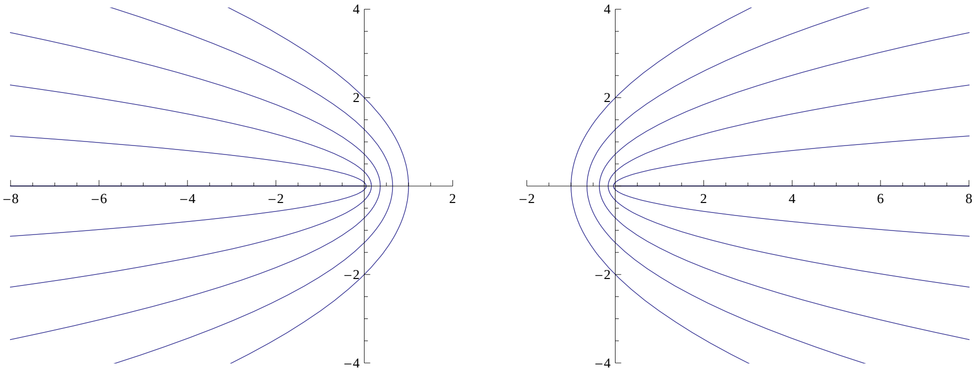
We study functions $f : \mathbb{C} \supset D \to \mathbb{C}$, where D is (usually) a domain.

Question: How to "visualize" functions $f : \mathbb{C} \supset D \to \mathbb{C}$?

A common approach is to investigate the images/pre-images of "simple" curves in the domain or image plane.

Example. $f(z) = z^2$ maps the vertical and horizontal lines to parabolas (or rays, if $c = 0$):

$$c + it \mapsto (c^2 - t^2) + 2ict \quad \text{and} \quad t + ic \mapsto (t^2 - c^2) + 2ict.$$



(ex) Show that a *Möbius transformation*[2] $z \mapsto \frac{az+b}{cz+d}$, $ad - bc \neq 0$, maps lines and circles to lines and circles.
Hint: $\alpha|z|^2 + \beta(z+\overline{z}) + i\gamma(z-\overline{z}) + \delta = 0$ is the equation of a circle or a line in $\mathbb{C}$; investigate $f(z) = \frac{1}{z}$ and write $\frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c}\frac{1}{cz+d}$.

Definition. $f : \mathbb{C} \supset D \to \mathbb{C}$ is *continuous (on D)* if, for all $z_0 \in D$,

$$\lim_{z \to z_0} f(z) = f(z_0) \qquad \Leftrightarrow \qquad \forall \varepsilon > 0\ \exists \delta > 0 : z \in B_\delta(z_0) \Rightarrow f(z) \in B_\varepsilon(f(z_0)).$$

Remark. In this definition $z \to z_0$ on *any* path.

---

[1] The theorems in this section are slightly different from the ones in [DET] since there $\mathbb{C} \subset \widehat{\mathbb{C}}$ equipped with the chordal metric is considered.

[2] A really nice video about Möbius transformations is available at

`http://www.youtube.com/watch?v=JX3VmDgiFnY&NR=1`

(ex) Show that

$$\mathbb{C}\setminus\{0\} \ni z = x+iy \mapsto f(z) = \frac{x^2y}{x^4+y^2} \in \mathbb{R} \subset \mathbb{C}$$

has no continuous extension to the whole plane.

Theorem II.1.1. *If $f : \mathbb{C} \supset K \to \mathbb{C}$ is continuous on a compact set K then f is bounded.*

Remark. It makes no sense any more to talk about a maximum of f (note that $\mathbb{C}$ is not *ordered*). So, the M11 proof has to be modified.

(ex) Let $f : \mathbb{C} \supset D \to \mathbb{C}$ be continuous and $K \subset D$ compact. Show that $f(K) \subset \mathbb{C}$ is compact and, in particular, bounded.

Definition. $f : \mathbb{C} \supset K \to \mathbb{C}$ is called *uniformly continuous (on K)* if

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall z, z' \in K : |z - z'| < \delta \Rightarrow |f(z) - f(z')| < \varepsilon.$$

Theorem II.1.2. *If $f : \mathbb{C} \supset K \to \mathbb{C}$ is continuous on a compact set K then f is uniformly continuous on K.*

Proof. . . . as in M11 (ex). ☐

II.2. Differentiability

References: [DET, Section 2.2] and [ST, Section 4.1]

Here comes the main definition:

Definition. A function $f : \mathbb{C} \supset D \to \mathbb{C}$ on a domain D is called *(complex) differentiable at $z_0 \in D$* if

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists.

If f is differentiable for every $z \in D$ then it is called *(complex) differentiable on D*, *(complex) analytic* or *holomorphic.*

Example. (i) $f(z) \equiv c$ is holomorphic on $\mathbb{C}$ with $f' \equiv 0$.

(ii) $f(z) = z$ is holomorphic on $\mathbb{C}$ with $f'(z) \equiv 1$.

(iii) $f(z) = \overline{z}$ is nowhere complex differentiable.

$$\lim_{h\to 0} \frac{\overline{z+h} - \overline{z}}{h} = \lim_{h\to 0} \frac{\overline{h}}{h} = \begin{cases} 1 & \text{if } h \text{ is real,} \\ -1 & \text{if } h \text{ is imaginary,} \\ \dots \end{cases}$$

(iv) Similarly, $f(z) = \operatorname{Re} z$, $f(z) = \operatorname{Im} z$ and $f(z) = |z|$ are not differentiable anywhere in $\mathbb{C}$.

ex Show that $\mathbb{C} \ni z \mapsto f(z) = |z|^2$ is differentiable at $z = 0$ only.

Theorem II.2.1. *If f is differentiable at z, then f continuous at z.*

Proof. …as in M11 ex. $\square$

Theorem II.2.2 (Algebra of differentiable functions)**.** *If $c \in \mathbb{C}$ and f, g are differentiable at z then so are*

$$\text{(i)} \quad cf, \qquad \text{(ii)} \quad f+g, \qquad \text{(iii)} \quad fg, \qquad \text{(iv)} \quad \frac{f}{g} \quad \text{if } g(z) \neq 0$$

with the usual derivatives.

Proof. …as in M11 ex. $\square$

Remark. Using the above formula, we see that every polynomial is holomorphic on $\mathbb{C}$, and every rational function (the quotient of two polynomials) is holomorphic outside the roots of its denominator.

Theorem II.2.3 (Chain rule)**.** *If f is differentiable at z and g is differentiable at $f(z)$, then $g \circ f$ is differentiable at z with $(g \circ f)'(z) = g'(f(z)) \cdot f'(z)$.*

Proof. …as in M11 ex. $\square$

II.3. The Cauchy-Riemann Equations

Reference: [DET, Section 2.3] and [ST, Sections 4.2 & 4.3]
(Necessary and sufficient Cauchy-Riemann conditions: [DET, Theorems 2.3.1 & 2.3.2] and [ST, Proposition 4.4 & Theorem 4.6])

Here we think of $\mathbb{C} \cong \mathbb{R}^2$ and write

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

in terms of its real and imaginary parts $u, v : \mathbb{R}^2 \supset D \to \mathbb{R}$.

Recall. A map $f : \mathbb{R}^n \supset U \to \mathbb{R}^m$ is *differentiable at $p \in U$*, if there is a linear map $\mathrm{d}f_p : \mathbb{R}^n \to \mathbb{R}^m$ so that

$$\lim_{h \to 0} \frac{\|f(p+h) - f(p) - \mathrm{d}f_p(h)\|}{\|h\|} = 0.$$

Given the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$, the limits (if they exist)

$$\frac{\partial f}{\partial x_i}(p) = \lim_{t \to 0} \frac{f(p+te_i) - f(p)}{t}$$

are the *partial derivatives of f at p.*

Remark.

- $\exists\, \mathrm{d}f_p \;\Rightarrow\; \exists\, \frac{\partial f}{\partial x_i}(p)$: if f is differentiable at p, then all partial derivatives exist and $\mathrm{d}f_p$ is given by the *Jacobi matrix* (whose columns are the partial derivatives).
- $\exists\, \frac{\partial f}{\partial x_i}(p) \;\not\Rightarrow\; \exists\, \mathrm{d}f_p$: the existence of the partial derivatives at p is not sufficient for f to be differentiable at p (even though its Jacobi matrix exists); example ⓔⓧ:

$$\mathbb{R}^2 \ni (x,y) \mapsto \frac{xy}{\sqrt{x^2+y^2}} \in \mathbb{R}.$$

- $\exists\, \frac{\partial f}{\partial x_i} \in C^0(U) \;\Rightarrow\; \forall p \in U\, \exists\, \mathrm{d}f_p$: if all partial derivatives exist and are continuous then f is (continuously) differentiable.
- $\exists\, \mathrm{d}f_p \;\not\Rightarrow\; \exists\, \frac{\partial f}{\partial x_i} \in C^0(U)$: If f is differentiable, then the partial derivatives might be discontinuous[3].

Informally, we see that real analysis is “complicated”!

Remark and Examples. We interpret the (real) differentiation geometrically:

- Let $f : \mathbb{R} \to \mathbb{R}$. Then, for small $x - x_0$, we have[4] $f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0)$. Here, we can interpret $f'(x_0)$ either as a “local scale factor” (by which $(x - x_0)$ is scaled), or we use the more familiar interpretation that $f(x_0) + f'(x_0) \cdot (x - x_0)$ is the “best possible linear approximation” of $f(x)$ in x_0.
- Let $f : \mathbb{R}^2 \to \mathbb{R}^2$. Then we have $f(x,y) \approx f(x_0,y_0) + \mathrm{d}f_{(x_0,y_0)}(x - x_0, y - y_0)^t$. Here, the Jacobi matrix $\mathrm{d}f_{(x_0,y_0)} \in M(2 \times 2, \mathbb{R})$ wherefore the derivative is locally an affine map, i.e., $f(x_0,y_0) + \mathrm{d}f_{(x_0,y_0)}(x - x_0, y - y_0)^t$ is the “best possible linear approximation” of $f(x,y)$ in (x_0,y_0).

Now , consider a linear transformation $(x,y) \mapsto A(x,y)^t$ given by some matrix $A \in M(2 \times 2, \mathbb{R})$ has the following effect:

- The standard basis vectors e_1 and e_2 are mapped to the first and the second column of A.
- A circle is mapped to an ellipse.

We would like to look at some examples of derivatives of functions $\mathbb{R}^2 \to \mathbb{R}^2$. These, however, are 2×2-matrices! So, to get a geometric intuition what the derivative is, we look at its effect on the basis vectors and a unit circle.

We consider the following three maps $f_k : \mathbb{R}^2 \to \mathbb{R}^2$: $f_1(x,y) = (x^2 - xy, 2xy)$, $f_2(x,y) = (x^2 - y^2, 2xy)$ and $f_3(x,y) = (x^2 - xy, -2xy)$.

---

[3] Consider

$$f(x,y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y} & \text{if } x \cdot y \neq 0, \\ x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \text{ and } y = 0, \\ y^2 \sin\frac{1}{y} & \text{if } x = 0 \text{ and } y \neq 0, \\ 0 & \text{if } x = 0 = y. \end{cases}$$

Then, the partial derivatives f_x, f_y are discontinuous at the orgin, but f is differentiable everywhere ⓔⓧ.

[4] To line it up with the definition of differentiability at the beginning of this section, we have $f(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + r(x - x_0)$ with $\lim_{h \to 0} \frac{r(h)}{h} = 0$, so

We calculate the partial derivatives of f_1: $\partial_x f_1$ is given by

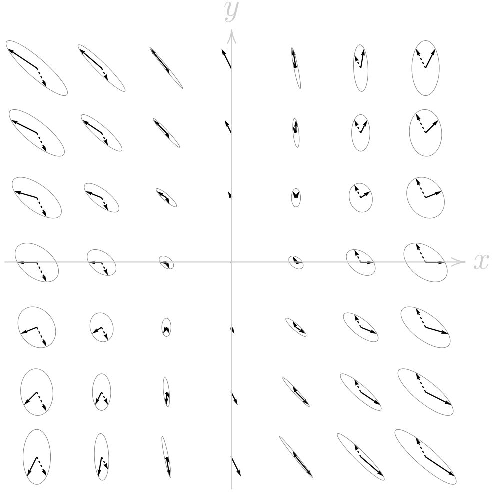
$$\frac{\partial f_1}{\partial x}(x_0, y_0) = \lim_{t\to 0} \frac{\begin{pmatrix} ((x_0+t)^2 - (x_0+t)y_0) - (x_0^2 - x_0 y_0) \\ (2(x_0+t)y_0) - (2x_0 y_0) \end{pmatrix}}{t} = \begin{pmatrix} 2x_0 - y_0 \\ 2y_0 \end{pmatrix}.$$

A similar calculation yields $\frac{\partial f_1}{\partial y}(x_0, y_0)$ and we obtain the following Jacobi matrix:

$$(\mathrm{d}f_1)_{(x_0,y_0)} = \begin{pmatrix} 2x_0 - y_0 & -x_0 \\ 2y_0 & 2x_0 \end{pmatrix}$$

Note that both partial derivatives are continuous, thus by the previous remark this Jacobi matrix is indeed the derivative of f_1.

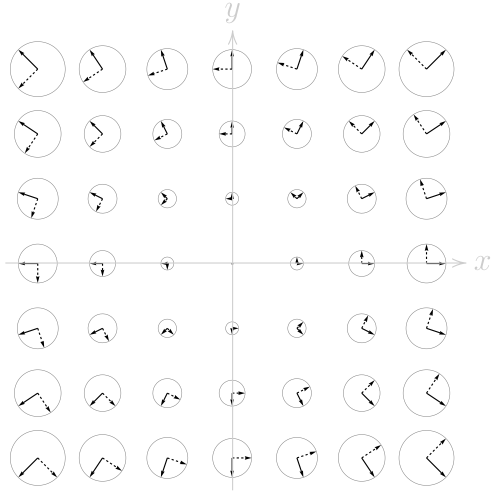
Now we study what this Jacobi matrix "geometrically does": we look at the effect of $(\mathrm{d}f_1)_{(x_0,y_0)}$ on the the standard basis vectors $e_1 = (1,0)^t$, $e_2 = (0,1)^t$ and the unit circle. This results in the picture on the right: Centred at the point (x_0, y_0), we attach the corresponding ellipse (the image of the unit circle under the Jacobian) and the image of e_1 (solid arrow) and e_2 (dashed arrow).



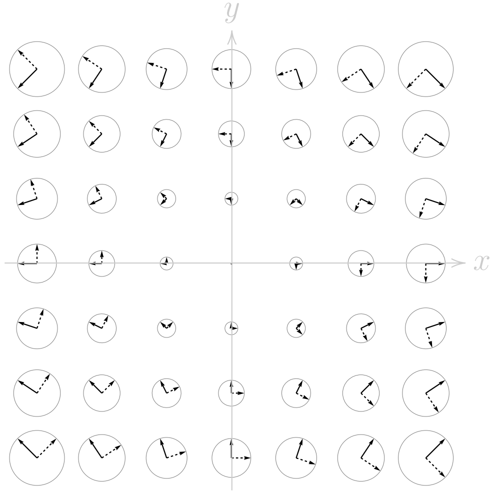
"Geometry" of $\mathrm{d}f_1$.

For f_2 and f_3 we calculate the following Jacobians:

$$(\mathrm{d}f_2)_{(x_0,y_0)} = \begin{pmatrix} 2x_0 & -2y_0 \\ 2y_0 & 2x_0 \end{pmatrix} \qquad \text{and} \qquad (\mathrm{d}f_3)_{(x_0,y_0)} = \begin{pmatrix} 2x_0 & 2y_0 \\ -2y_0 & 2x_0 \end{pmatrix}$$



"Geometry" of $\mathrm{d}f_2$.



"Geometry" of $\mathrm{d}f_3$.

For holomorphic functions, pictures like that for $\mathrm{d}f_1$ (with ellipses and varying angles between the images of the basis vectors) or for $\mathrm{d}f_3$ (where the "handiness" of the basis vectors changes when considering their image) cannot occur (accidently, $f_2(z) = z^2$ is holomorphic), as the following statement shows which connectes real and complex diffeerentiation.

Theorem II.3.1 (Necessary Cauchy-Riemann conditions)**.** *If $f : \mathbb{C} \supset D \to \mathbb{C}$ is holomorphic then all first partial derivatives u_x, u_y, v_x, v_y of $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ exist and satisfy the so-called* Cauchy-Riemann equations

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x.$$

Moreover, $f' = u_x + iv_x = v_y - iu_y$.

Proof. (ex) Compute $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ twice, for $h = t \in \mathbb{R}$ and for $h = it$ purely imaginary. ☐

(ex) Show that (the continuous extension of) $f(z) = \frac{z^5}{|z|^4}$ is not (complex) differentiable at $z = 0$ but satisfies the Cauchy-Riemann equations there. Can you find more functions with this property?

(ex) (i) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show: f is differentiable everywhere but f' is not continuous.

(ii) Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2 \sin(1/z)$ for $z \neq 0$, $f(0) = 0$. The solution to part (i) almost seems to give a proof that f is (complex) differentiable everywhere but f' is not continuous at the origin. This is impossible – where does the "proof" fail?

Geometric meaning of the Cauchy-Riemann equations. Thinking of f as a map $(u, v) : \mathbb{R}^2 \supset D \to \mathbb{R}^2$, the derivative of f at $z = (x, y)$ is its Jacobi matrix

$$f'(z) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\Bigg|_{(x,y)}.$$

The Cauchy-Riemann equations say that $f'(z)$ is a complex number $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ (see Section I.1), i.e., a stretch-rotation.

Another way to put the Cauchy-Riemann equations is to say that the derivative $f'(z)$ is *complex linear*; since $f'(z) : \mathbb{R}^2 \to \mathbb{R}^2$ is already real linear it suffices to verify that it commutes (ex) with $i \simeq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(ex) Instead of writing a mapping in terms of its real and imaginary parts (i.e., $f = u + iv$), it is sometimes more convenient to write it in terms of modulus and argument:

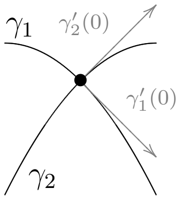
$$f(z) = f(x + iy) = R(x, y)\ (\cos \Psi(x, y) + i \sin \Psi(x, y)).$$

Show that in this case the Cauchy-Riemann equations are

$$\partial_x R = R \cdot \partial_y \Psi \qquad \text{and} \qquad \partial_y R = -R \cdot \partial_x \Psi.$$

Corollary II.3.2. *A holomorphic map $f : D \to \mathbb{C}$ is* conformal, *i.e., preserves the intersection angle of curves in D.*

Proof. We take two curves $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to D$ with $\gamma_1(0) = z = \gamma_2(0)$ and $|\gamma_1'(0)| = 1 = |\gamma_2'(0)|$.
Then $\gamma_i'(0) = \cos\varphi_i + i \sin\varphi_i$ and the (oriented) intersection angle of the two curves is $\alpha = \varphi_1 - \varphi_2$, which is given by $\frac{\gamma_1'(0)}{\gamma_2'(0)} = \cos\alpha + i\sin\alpha$.



Now consider $f \circ \gamma_i$ and compute

$$\frac{(f\circ\gamma_1)'(0)}{(f\circ\gamma_2)'(0)} = \frac{f'(z)\cdot\gamma_1'(0)}{f'(z)\cdot\gamma_2'(0)} = \cos\alpha + i\sin\alpha$$

to see that the intersection angle of the $f \circ \gamma_i$ is the same as the one of the γ_i. ☐

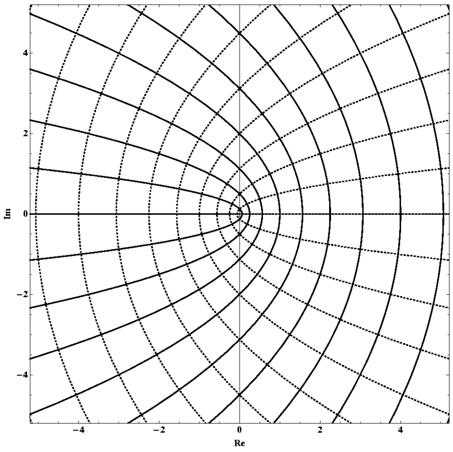


Image of vertical (solid) and horizontal (dotted) lines under the map $z \mapsto z^2$. These parabolas intersect each other perpendicular.

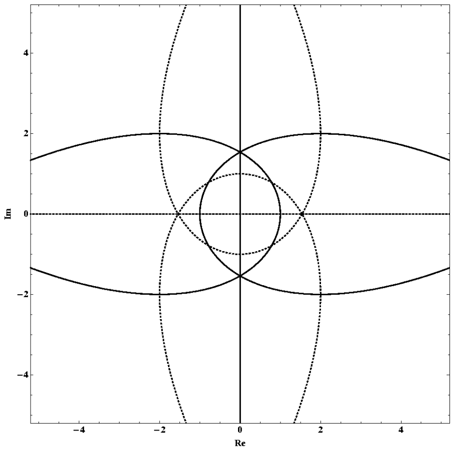
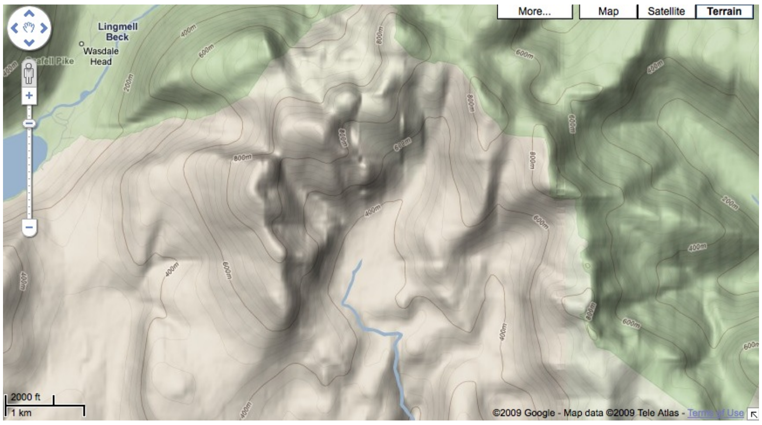


Image of a few vertical (solid) and horizontal (dotted) lines under the map $f(z) = z^3$. A vertical line $t \mapsto x_0 + i\,t$ intersects a horizontal line $t \mapsto t + i\,y_0$ perpendicularly at $x_0 + i\,y_0$, and so do their images at $f(x_0 + i\,y_0)$. However, they might intersect at other places arbitrarily.

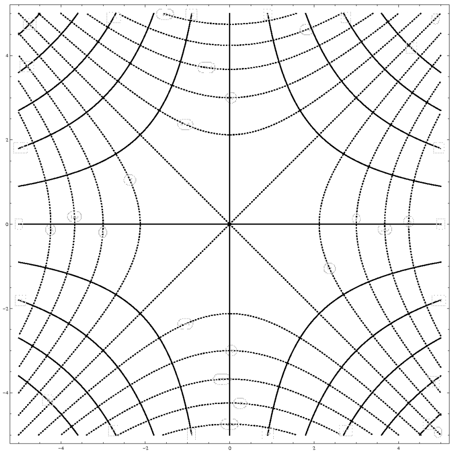
ex Let $f(x+iy) = u(x,y) + i\,v(x,y)$ be holomorphic in a domain D, and let $(x_0, y_0) \in D$ where the gradient vectors of u and v do not vanish. Set $u_0 = u(x_0,y_0)$ and $v_0 = v(x_0,y_0)$. Show that the *level curves* $u(x,y) = u_0$ and $v(x,y) = v_0$ intersect perpendicularly at (x_0, y_0).

Show a similar result if one uses $f(x+iy) = R(x,y)\ \mathrm{e}^{i\Psi(x,y)}$.

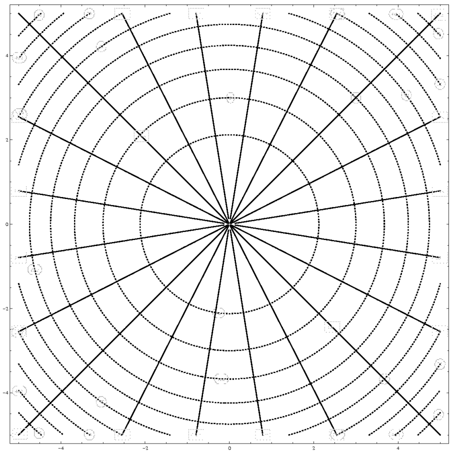
In plain words, this says that level curves of constant real and imaginary part (or of constant modulus and argument) are perpendicular to each other.



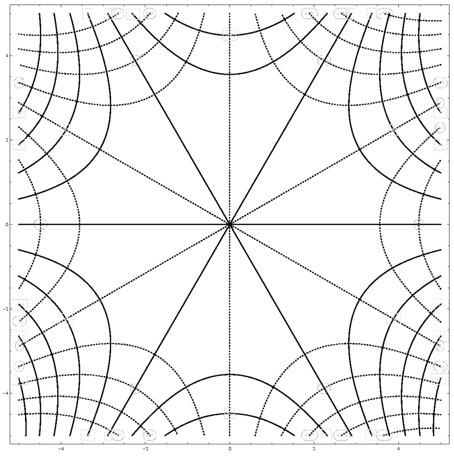
Some "real-world" level curves, the lines of the same height above sea-level, as seen on `http://maps.google.co.uk`. We are looking at a region in the Lake District here.



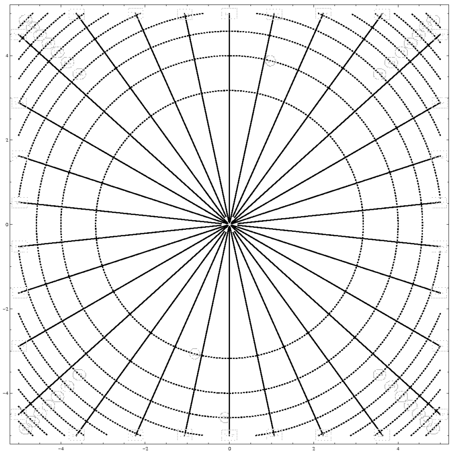
For $f(z) = z^2$, level curves of constant real (dotted) and imaginary (solid) part intersect perpendicularly.



For $f(z) = z^2$, level curves of constant modulus (dotted) and argument (solid) intersect perpendicularly.



For $f(z) = z^3$, level curves of constant real (dotted) and imaginary (solid) part intersect perpendicularly.



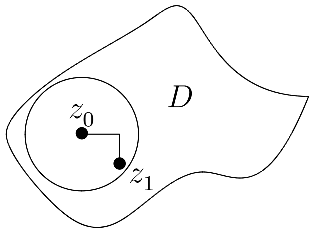
For $f(z) = z^3$, level curves of constant modulus (dotted) and argument (solid) intersect perpendicularly.

Corollary II.3.3. *Let $f : \mathbb{C} \supset D \to \mathbb{C}$ be holomorphic with $f' \equiv 0$. Then f is constant.*

Proof. We first show[5]: If f is *locally constant*[6] at each point of D, then f is constant. For this, choose $z_0 \in D$ and consider $A_1 = \{z \in D \mid f(z) = f(z_0)\}$ and $A_2 = D \setminus A_1$. The set A_2 is the inverse image/preimage of the open set $\{w \in \mathbb{C} \mid w \neq f(z_0)\}$ and therefore open (by the continuity of f). Therefore, A_1 is closed in D. The set A_1 is also open: If $z_1 \in A_1$, then there is some open disk $B_\varepsilon(z_1) \subset A_1$ (as f is locally constant at z_1). Using Lemma I.5.2 establishes $A_1 = D$.

Now, $f' = u_x + i\, v_x = v_y - i\, u_y = 0$ for all $z \in D$, and so $u_x = u_y = v_x = v_y = 0$.

Let $z_0 = x_0 + iy_0 \in D$, let $B_\varepsilon(z_0) \subset D$ and choose $z_1 = x_1 + iy_1 \in B_\varepsilon(z_0)$. Since $\partial_x u(x, y_0) = 0 = \partial_x v(x, y_0)$, the functions $x \mapsto u(x, y_0)$ and $x \mapsto v(x, y_0)$ are constant on the interval $[x_0, x_1]$ (or $[x_1, x_0]$ if $x_1 \leq x_0$), wherefore



$$\begin{aligned} f(x_1 + iy_0) &= u(x_1, y_0) + i\, v(x_1, y_0) \\ &= u(x_0, y_0) + i\, v(x_0, y_0) \\ &= f(x_0 + iy_0). \end{aligned}$$

Similarly, since $\partial_y u(x, y_0) = 0 = \partial_y v(x, y_0)$, the functions $y \mapsto u(x_1, y)$ and $y \mapsto v(x_1, y)$ are constant on the interval $[y_0, y_1]$. Thus

$$f(x_1 + iy_1) = u(x_1, y_1) + i\, v(x_1, y_1) = u(x_1, y_0) + i\, v(x_1, y_0) = f(x_1 + iy_0) \;=\; f(x_0 + iy_0).$$

So, f is constant on the disk $B_\varepsilon(z_0)$. ☐

ⓔⓧ Let $f : \mathbb{C} \supset D \to \mathbb{C}$ be holomorphic. Prove that f is constant as soon as any one of $\operatorname{Re} f$, $\operatorname{Im} f$ or $|f|$ is constant.

Corollary II.3.4. *Let $f = u + iv : \mathbb{C} \supset D \to \mathbb{C}$ be holomorphic. Then $u, v : D \to \mathbb{R}$ are* harmonic functions*, i.e.,*

$$\Delta u = u_{xx} + u_{yy} = 0 \qquad \text{and} \qquad \Delta v = 0.$$

Since $\nabla v = i \nabla u$, i.e., $\nabla v = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla u$, one says that v is the conjugate harmonic function *of u.*

Remark. In order to prove Corollary II.3.4 we need, at the moment, the additional assumption that the second partial derivatives of u and v exist and are continuous (so that Clairaut's theorem/Schwarz' lemma holds).

Later we will see that this already follows from holomorphicity.

ⓔⓧ Prove Corollary II.3.4 under the additional assumption that the second partial derivatives of $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ exist and are continuous.

---

[5] Alternatively, one might prove this corollary by taking $z, w \in D$ and a path $\gamma : [a, b] \to D$ joining them, and then applying the MVT (from M11) ⓔⓧ.

[6] We say that f is *locally contant* at z_0 if there is an $\varepsilon > 0$ s.t. $f(z) = f(z_0)$ for all $z \in B_\varepsilon(z_0) \subset D$.

Remark. Harmonic functions are studied in physics when considering fluid flows or electrical fields. We will have a brief look at this later in this course (see Section III.4.3).

Theorem II.3.5 (Sufficient Cauchy-Riemann conditions)**.** *If $u, v : \mathbb{R}^2 \supset D \to \mathbb{R}$ are continuously differentiable on D (have continuous first partial derivatives) which satisfy the* Cauchy-Riemann equations

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x,$$

then $f = u + iv$ is holomorphic on D.

Proof. Here we wheel out the definition of differentiability of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$: since f is (real) differentiable at $z = x + iy$

$$\begin{aligned} 0 &= \lim_{h\to 0} \frac{1}{|h|} |f(z+h) - f(z) - \mathrm{d}f_z(h)| \\ &= \lim_{h\to 0} \frac{1}{|h|} |f(z+h) - f(z) - (u_x h_1 + u_y h_2, v_x h_1 + v_y h_2)| \\ &\overset{\text{C-R}}{=} \lim_{h\to 0} \frac{1}{|h|} |f(z+h) - f(z) - (u_x h_1 - v_x h_2, v_x h_1 + u_x h_2)| \\ &= \lim_{h\to 0} \left| \frac{f(z+h) - f(z)}{h} - (u_x + iv_x) \frac{\cancel{h}}{\cancel{h}} \right|, \end{aligned}$$

where $\mathrm{d}f_z = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ is the differential of f at z.

Hence, f is (complex) differentiable at z, with $f'(z) = u_x(z) + iv_x(z)$. This holds for every $z \in D$, so f is holomorphic in D, as required. ☐

II.4. Examples

Reference: [DET, Section 2.5] and [ST, Chapter 5]

We look at the following examples:

(i) The exponential function $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto \mathrm{e}^z = \mathrm{e}^x(\cos y + i \sin y)$ is holomorphic $\textcircled{ex}$ with $f' = f$ and $f(z_1 + z_2) = f(z_1) f(z_2)$ (later, we will define the exponential function via its series expansion). Note that it is periodic with period $2\pi i$ (since $\cos, \sin : \mathbb{R} \to \mathbb{R}$ are 2π-periodic).

Note that $\mathrm{e}^z = \mathrm{e}^x(\cos y + i \sin y)$ maps horizontal lines to rays from $w = 0$ and vertical lines to concentric circles.

(ii) The (complex) trigonometric functions sin and cos are defined by

$$\cos z = \frac{\mathrm{e}^{iz} + \mathrm{e}^{-iz}}{2} \qquad \text{and} \qquad \sin z = \frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{2i}$$

for all $z \in \mathbb{C}$. They are holomorphic and have the expected properties (derivatives, $\cos^2 z + \sin^2 = 1$, 2π-periodicity, etc.).

(iii) The (complex) hyperbolic functions sinh and cosh are defined by

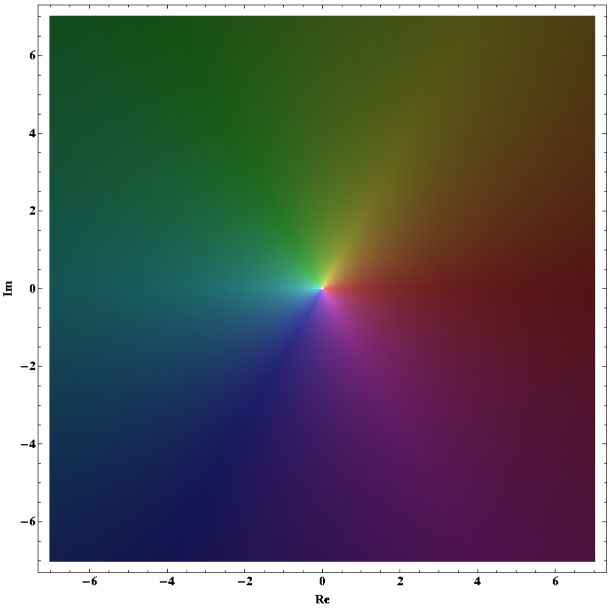
$$\cosh z = \frac{\mathrm{e}^z + \mathrm{e}^{-z}}{2} = \cos(iz) \qquad \text{and} \qquad \sinh z = \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{2} = -i\sin(iz)$$

for all $z \in \mathbb{C}$. They are also holomorphic.

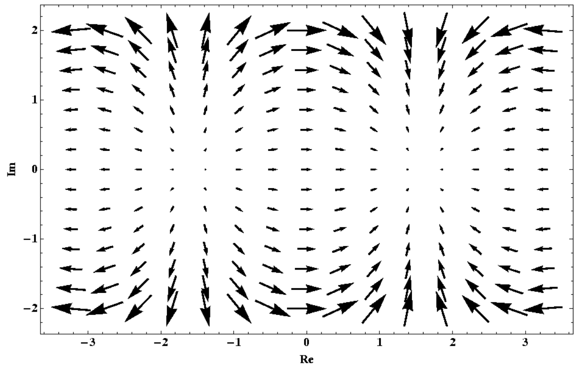
(iv) Riemann's Zeta Function[7] is holomorphic on $\mathbb{C} \setminus \{0\}$ as remarked on p. 5.

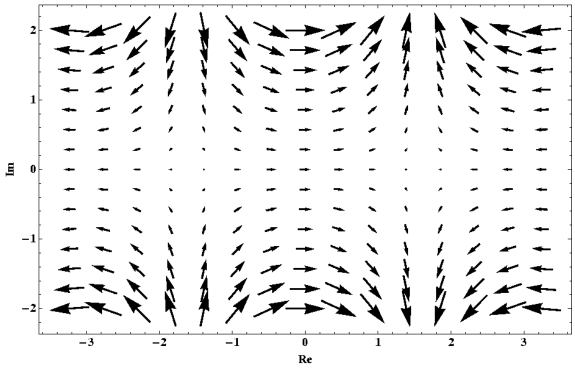
We will later also look at the complex logarithm and arbitrary powers.

We want to get some geometric intuition of these functions. Two methods we have already mentioned are to investigate the images/pre-images of "simple" curves (like vertical and horizontal lines) under the function in question, or the level curves of the function. If we now would like to see their "graph" in four-dimensional(!) space, we can refine the level curve method as follows: We use a colour palette to "visualise" two of the four dimensions. We colour the origin 0 white and colours get darker the greater the modulus of the complex number is (the point at infinity ∞ is black). A colour corresponds[8] to the argument of the complex number, e.g., the positive real numbers are red-ish. In the following, for a given function f, a point $z \in \mathbb{C}$ is coloured according to its value $f(z)$ by the above palette, e.g., if we have $f(z) = 1$ for a point z then that point z is coloured red.



A different method is to study vector fields: Instead of colouring the z according to $f(z)$, we attach the vector $f(z) = (\operatorname{Re} f(z), \operatorname{Im} f(z))$ to the point z. Below is the result if we do this for $f(z) = \cos(z)$ (on the left) and $f(z) = \overline{\cos(z)}$ (on the right).





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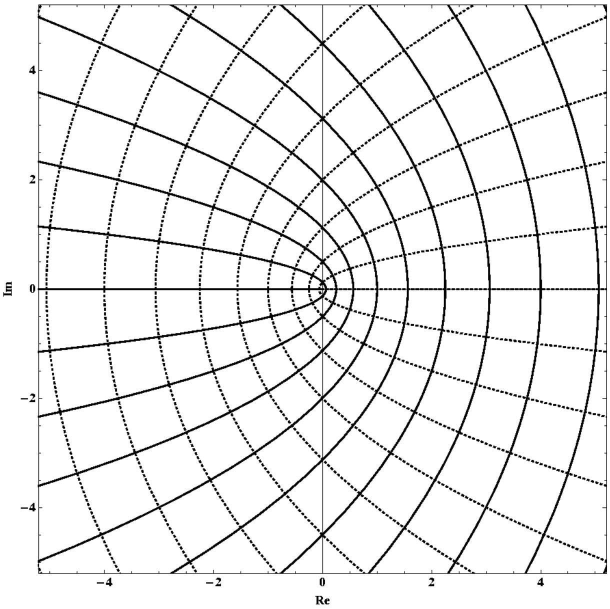
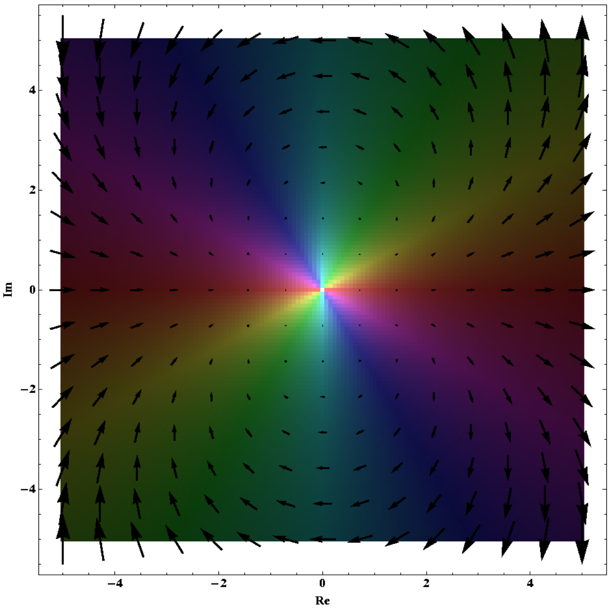
[7] An article that uses level curves to study Riemann's Zeta Function in the critical strip can be found at http://arxiv.org/abs/math.NT/0309433

[8] Compare this with the earlier discussed level sets/curves: the level curve of constant modulus corresponds to the line of the same brightness/darkness, while the level curve of constant argument corresponds to the line of constant colour (which might, however, become brighter or darker).

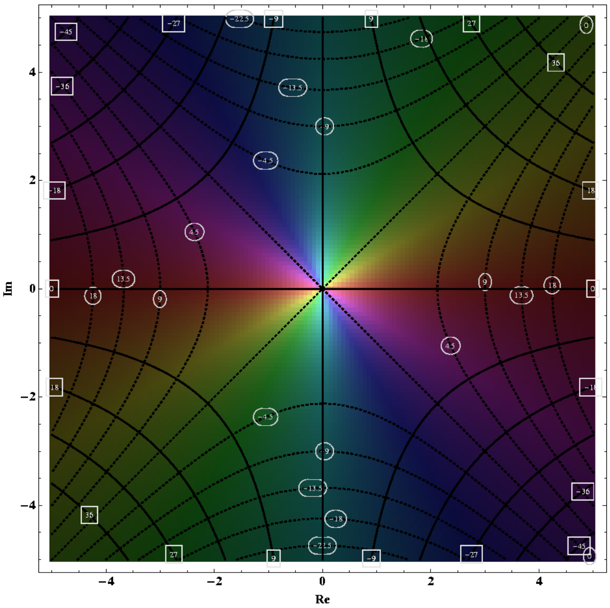
The function $z \longrightarrow z^2$:



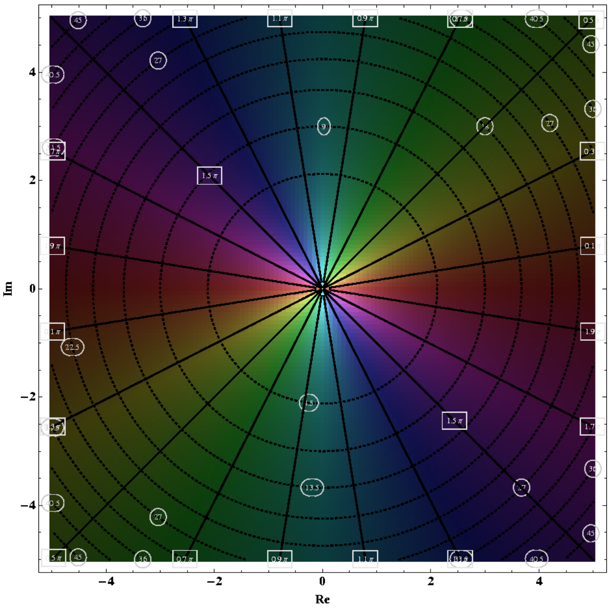
The image of the vertical (solid) and horizontal (dotted) lines under $f(z) = z^2$; they intersect perpendicularly.



Vector field $(\operatorname{Re} z^2, \operatorname{Im} z^2)$ and colouring of the complex plane for $z \mapsto z^2$.

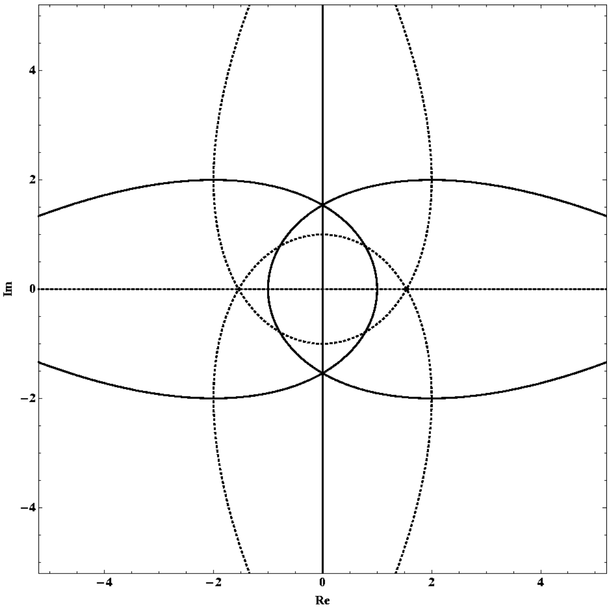


Colouring and level curves of constant real (dotted) and imaginary (solid) part for $z \mapsto z^2$; they intersect perpendicularly.

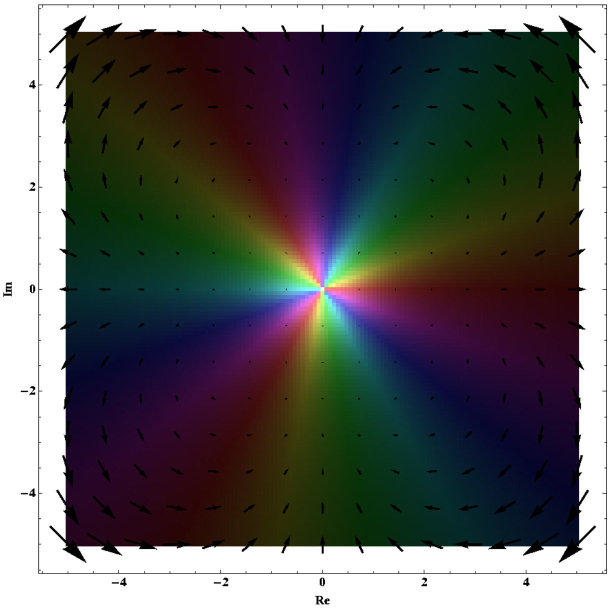


Colouring and level curves of constant modulus (dotted) and argument (solid) for $z \mapsto z^2$; they intersect perpendicularly.

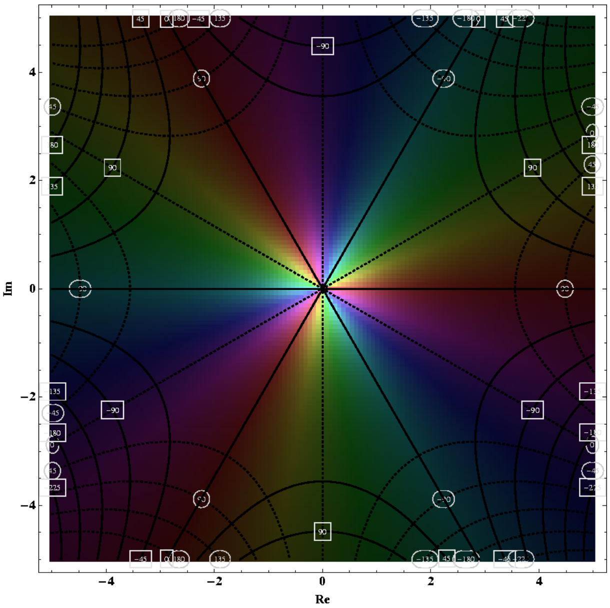
The function $z \mapsto z^3$:



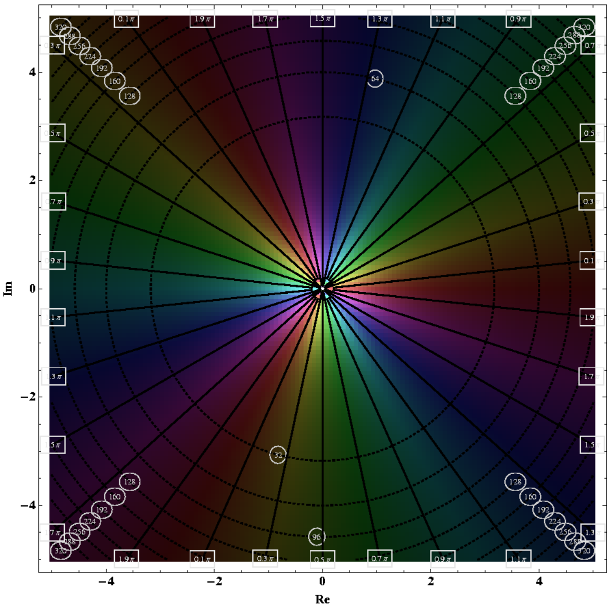
The image of the vertical (solid) and horizontal (dotted) lines under $f(z) = z^3$; they intersect perpendicularly.



Vector field $(\operatorname{Re} z^3, \operatorname{Im} z^3)$ and colouring of the complex plane for $z \mapsto z^3$.

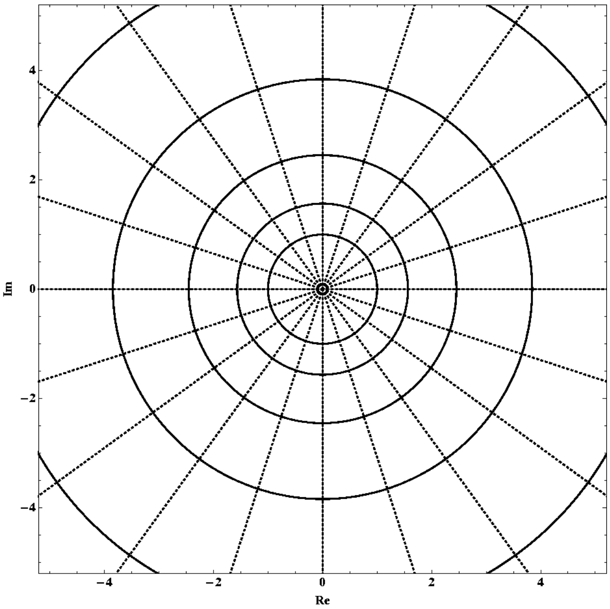


Colouring and level curves of constant real (dotted) and imaginary (solid) part for $z \mapsto z^3$; they intersect perpendicularly.

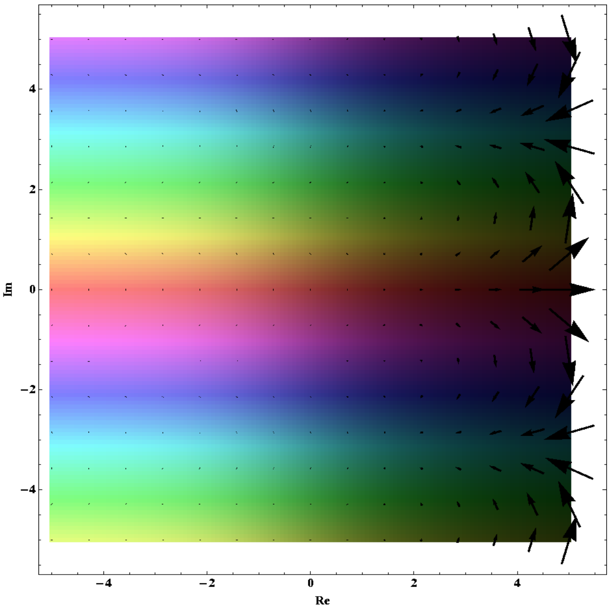


Colouring and level curves of constant modulus (dotted) and argument (solid) for $z \mapsto z^3$; they intersect perpendicularly.

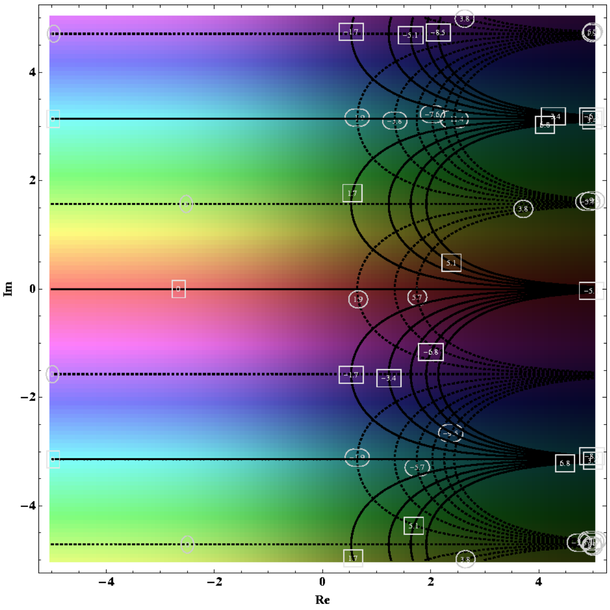
The complex exponential:



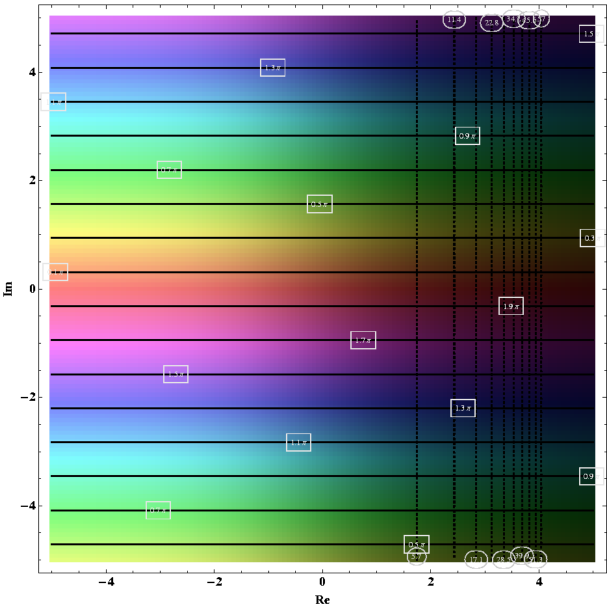
The image of the vertical (solid) and horizontal (dotted) lines under $f(z) = \exp z$; they intersect perpendicularly.



Vector field $(\operatorname{Re}\exp z, \operatorname{Im}\exp z)$ and colouring of the complex plane for $\exp z$.

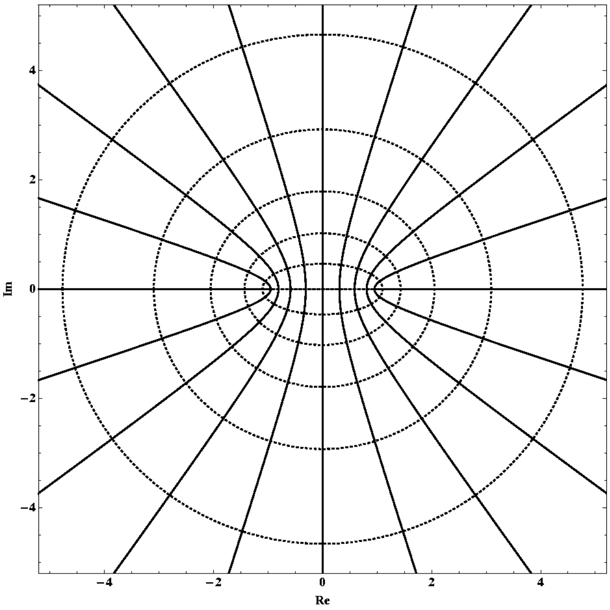


Colouring and level curves of constant real (dotted) and imaginary (solid) part for $\exp z$; they intersect perpendicularly.

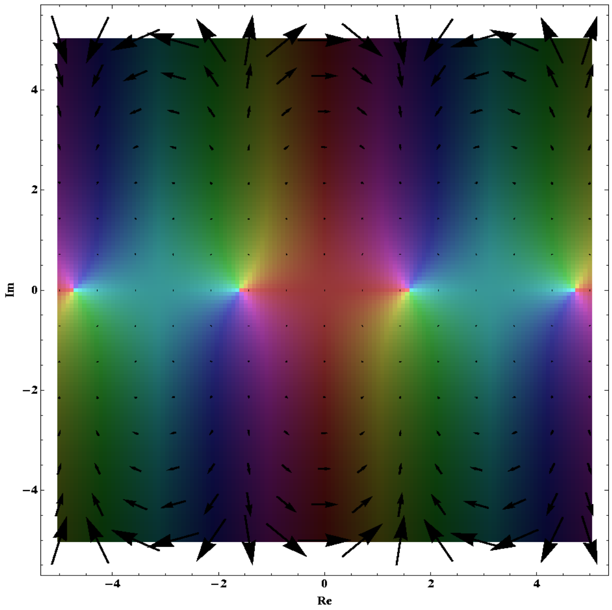


Colouring and level curves of constant modulus (dotted) and argument (solid) for $\exp z$; they intersect perpendicularly.

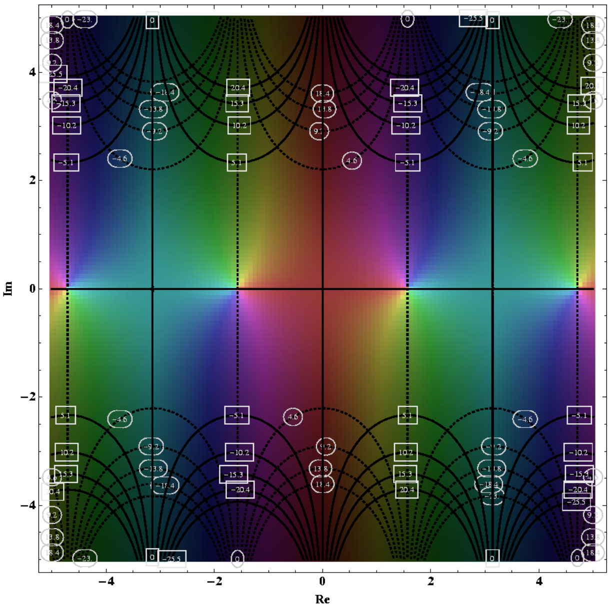
The complex cosine:



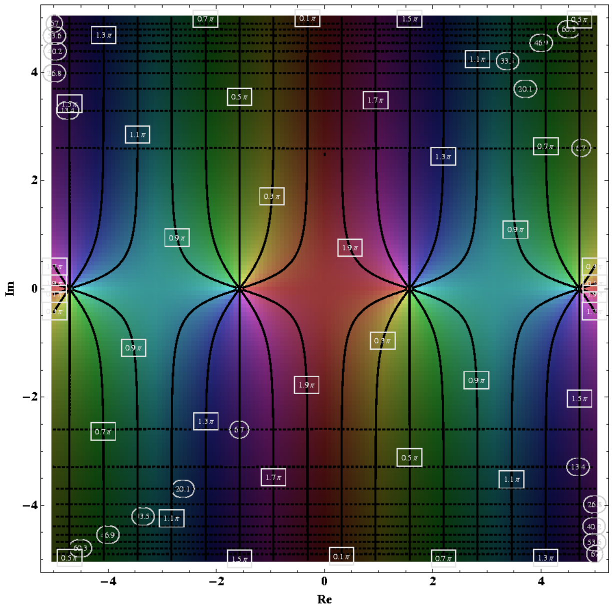
The image of the vertical (solid) and horizontal (dotted) lines under $f(z) = \cos z$; they intersect perpendicularly.



Vector field $(\operatorname{Re}\cos z, \operatorname{Im}\cos z)$ and colouring of the complex plane for $\cos z$.

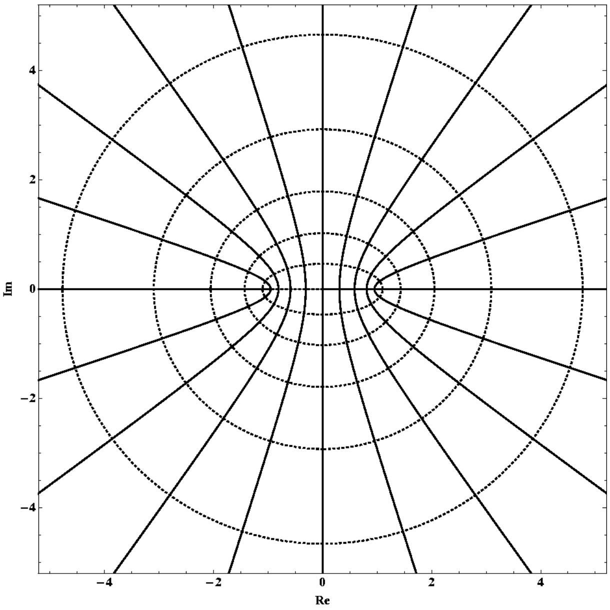


Colouring and level curves of constant real (dotted) and imaginary (solid) part for $\cos z$; they intersect perpendicularly.

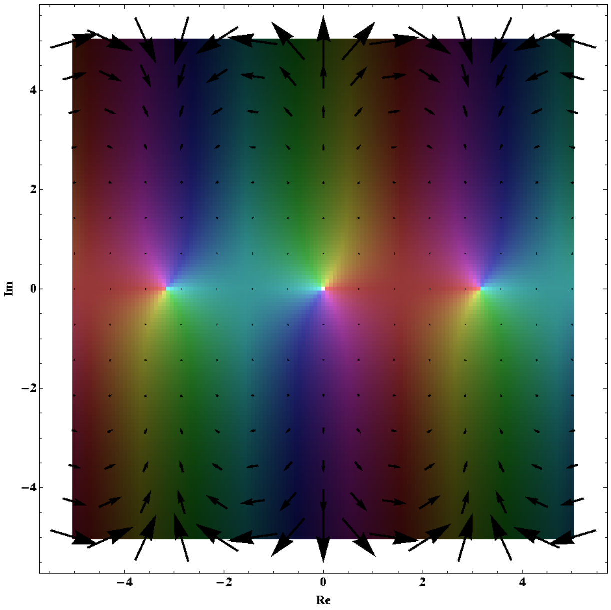


Colouring and level curves of constant modulus (dotted) and argument (solid) for $\cos z$; they intersect perpendicularly.

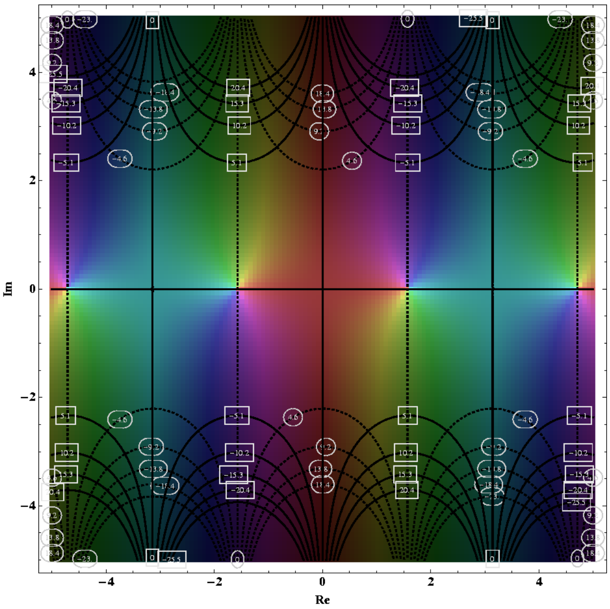
The complex sine:



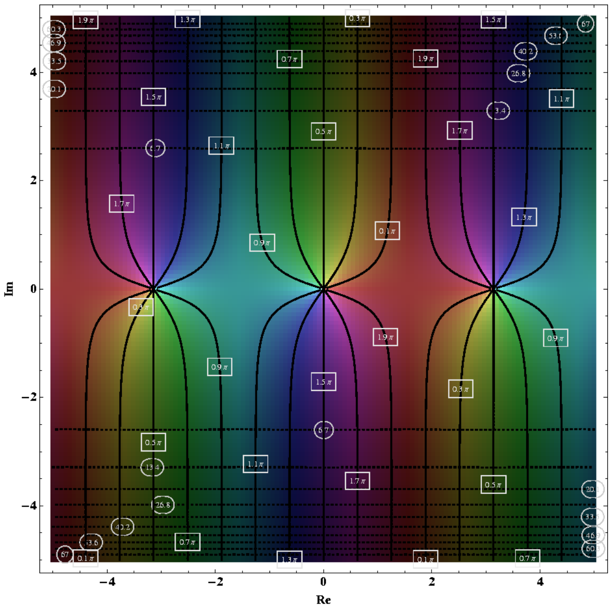
The image of the vertical (solid) and horizontal (dotted) lines under $f(z) = \sin z$; they intersect perpendicularly.



Vector field $(\operatorname{Re}\sin z, \operatorname{Im}\sin z)$ and colouring of the complex plane for $\sin z$.

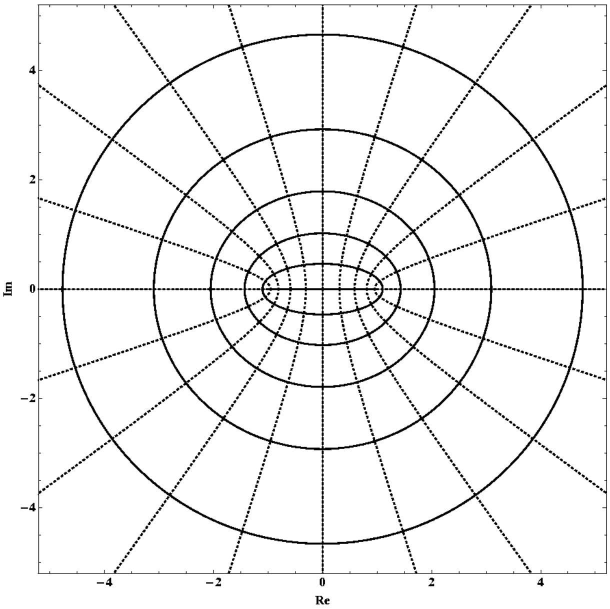


Colouring and level curves of constant real (dotted) and imaginary (solid) part for $\sin z$; they intersect perpendicularly.

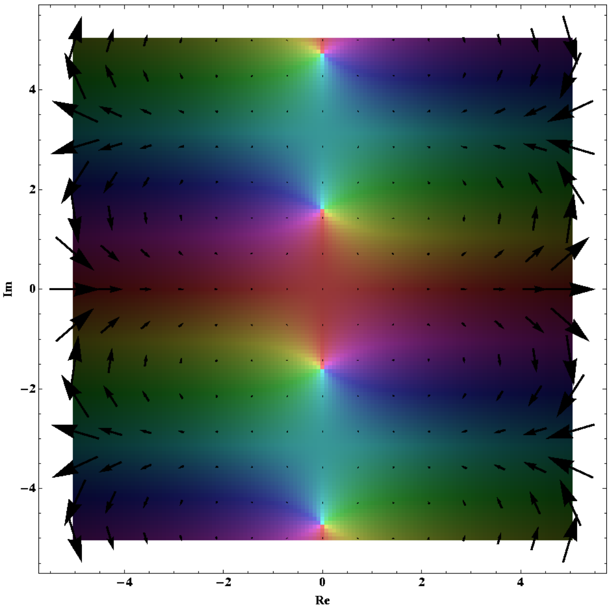


Colouring and level curves of constant modulus (dotted) and argument (solid) for $\sin z$; they intersect perpendicularly.

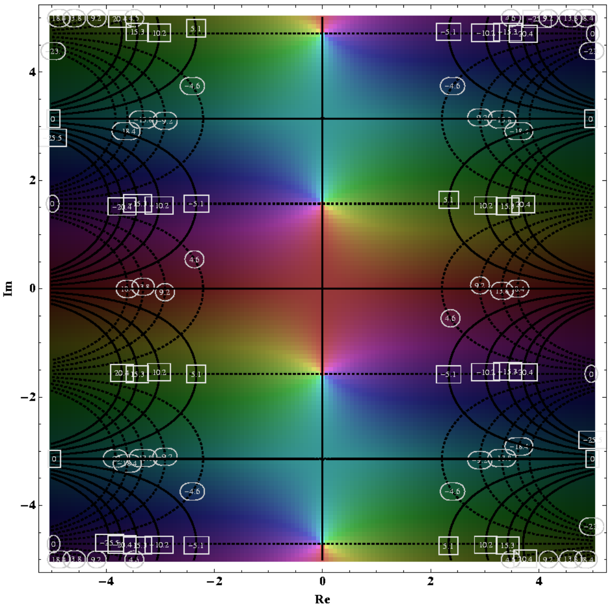
The complex hyperbolic cosine:



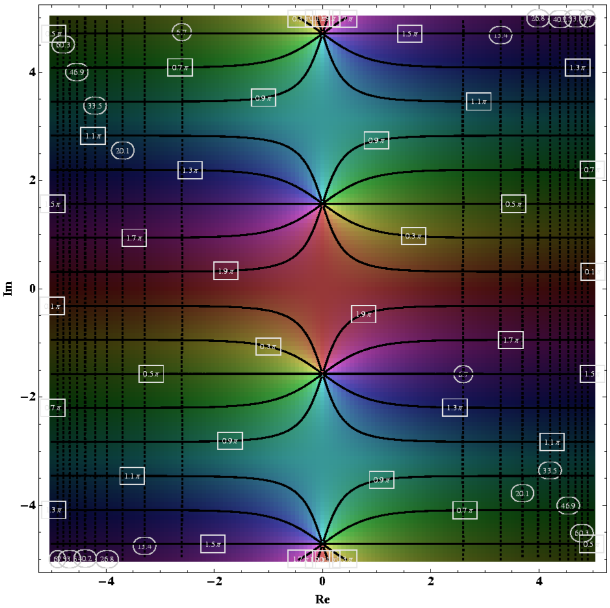
The image of the vertical (solid) and horizontal (dotted) lines under $f(z) = \cosh z$; they intersect perpendicularly.



Vector field $(\operatorname{Re}\cosh z, \operatorname{Im}\cosh z)$ and colouring of the complex plane for $\cosh z$.

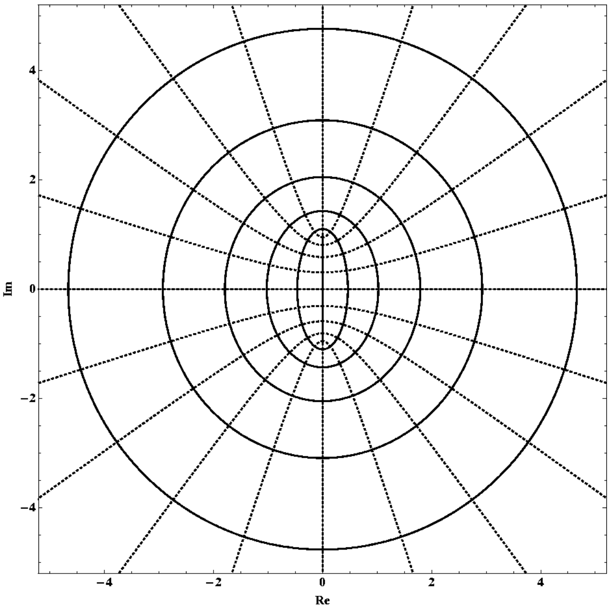


Colouring and level curves of constant real (dotted) and imaginary (solid) part for $\cosh z$; they intersect perpendicularly.

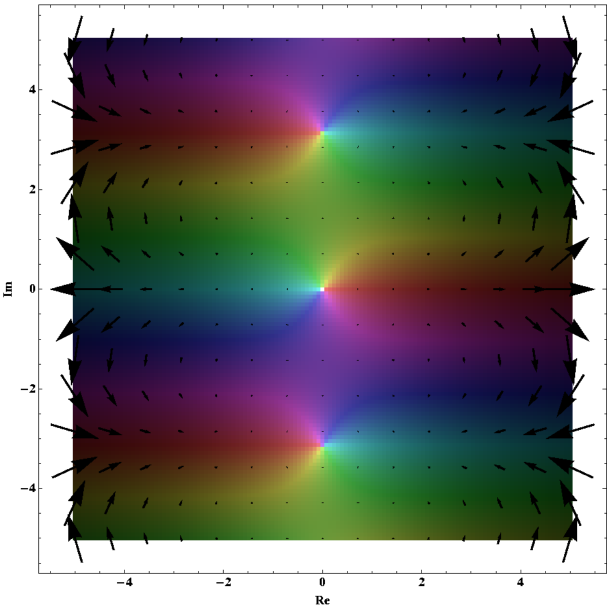


Colouring and level curves of constant modulus (dotted) and argument (solid) for $\cosh z$; they intersect perpendicularly.

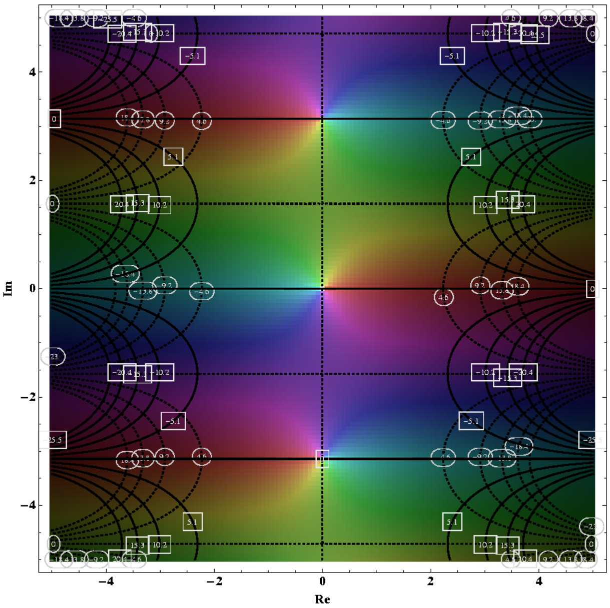
The complex hyperbolic sine:



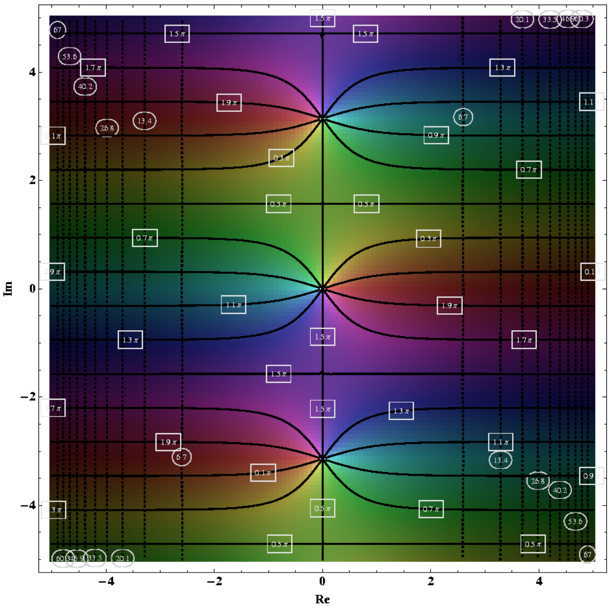
The image of the vertical (solid) and horizontal (dotted) lines under $f(z) = \sinh z$; they intersect perpendicularly.



Vector field $(\operatorname{Re}\sinh z, \operatorname{Im}\sinh z)$ and colouring of the complex plane for $\sinh z$.

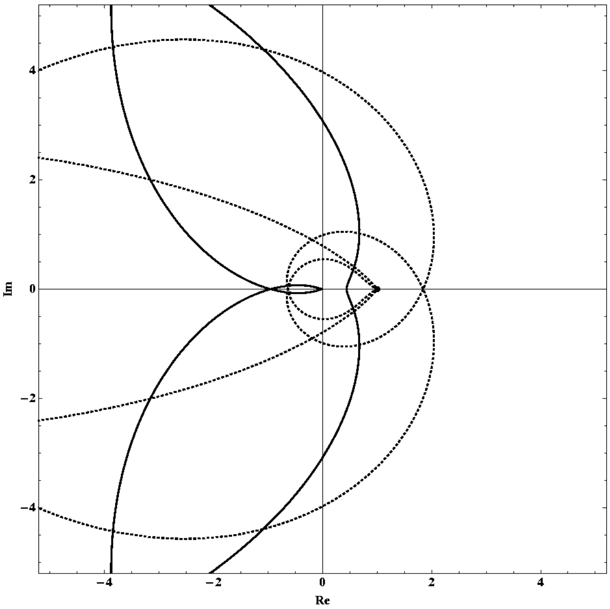


Colouring and level curves of constant real (dotted) and imaginary (solid) part for $\sinh z$; they intersect perpendicularly.

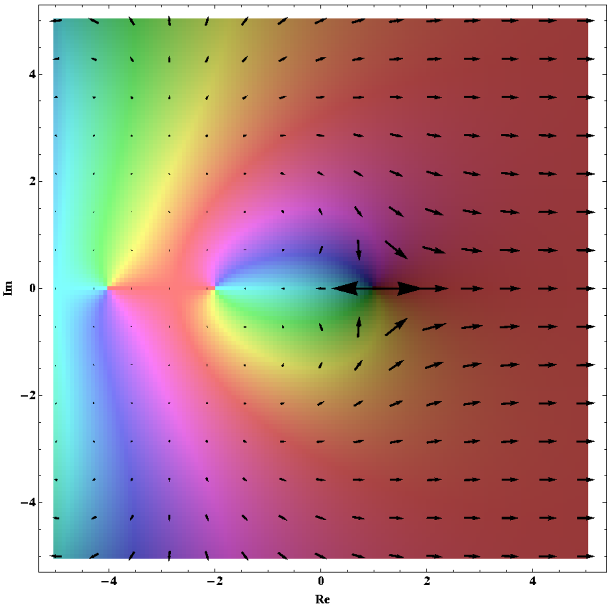


Colouring and level curves of constant modulus (dotted) and argument (solid) for $\sinh z$; they intersect perpendicularly.

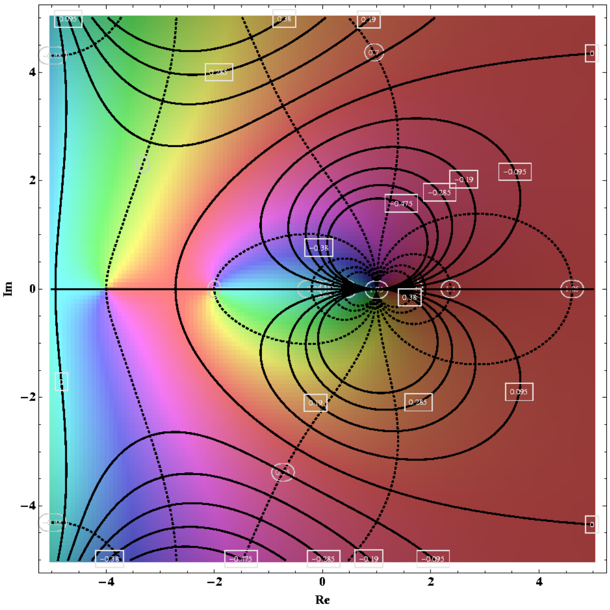
Riemann's Zeta Function:



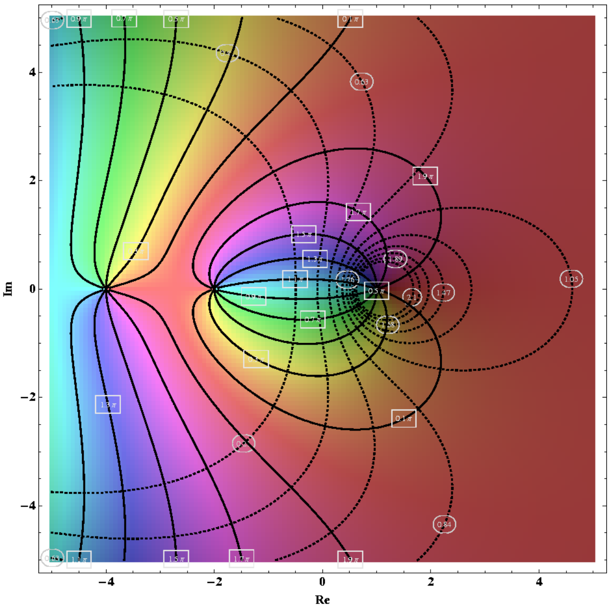
The image of the vertical (solid) and horizontal (dotted) lines under $f(z) = \zeta(z)$; they intersect perpendicularly.



Vector field $(\operatorname{Re}\zeta(z), \operatorname{Im}\zeta(z))$ and colouring of the complex plane for $\zeta(z)$.



Colouring and level curves of constant real (dotted) and imaginary (solid) part for $\zeta(z)$; they intersect perpendicularly.



Colouring and level curves of constant modulus (dotted) and argument (solid) for $\zeta(z)$; they intersect perpendicularly.

II.5. Cauchy-Riemann Revisited [9] (Not examinable!)

We will later see that a holomorphic function $f : D \to \mathbb{C}$ has derivatives of all orders; in particular, it thus has continuous partial derivatives u_x, u_y, v_x, v_y. Using Theorems II.3.1 & II.3.5 (necessary & sufficienten Cauchy-Riemann conditions), we can therefore state: $f : D \to \mathbb{C}$ is holomorphic iff $f = u + iv$ satiesfies the Cauchy-Riemann equations at every point of the domain D. In this case, $f' = u_x + i\,v_x$.

We can state all this using an alternate notation which is more suggestive, but needs some interpretation. We start with defining two operators $\partial/\partial z$ and $\partial/\partial \overline{z}$ by:

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\,\frac{\partial f}{\partial y}\right) &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial (iy)}\right) \\ \frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\,\frac{\partial f}{\partial y}\right) &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial (-iy)}\right) \end{aligned}$$

There are a few things to be said for this notation:

- The expressions we get "look right": One checks (writing $f = u + iv$) that f satisfies the Cauchy-Riemann equations iff
  $$\frac{\partial f}{\partial \overline{z}} = 0;$$
  in that case, the derivative is given by
  $$f' = \frac{\partial f}{\partial z}.$$
- The "partial derivatives" $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$ are often easy to calculate: Although the two operators $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$ are not really partial derivatives with respect to two independent variables (z and $\overline{z}$ are certainly not independent!), when can often pretend that they are and hence just easily "see" what they are. E.g., if f is written as polynomial in z and $\overline{z}$, then we can tell $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$ just by looking:
  $$\frac{\partial (z^3\overline{z}^2)}{\partial z} = 3z^2\overline{z}^2, \qquad \frac{\partial (z^3\overline{z}^2)}{\partial \overline{z}} = 2z^3\overline{z}.$$
  (Check that $\frac{\partial z}{\partial z} = 1 = \frac{\partial \overline{z}}{\partial \overline{z}}$ and $\frac{\partial z}{\partial \overline{z}} = 0 = \frac{\partial \overline{z}}{\partial z}$ and that the two operators satisfy the product rule.)

In view of the first point, one would like to say something like "f is holomorphic iff it does not have any $\overline{z}$'s in it"; however, that is too vague to be true (what about $f(z) = |z|^2 = z\overline{z}$?), but the next theorem will give a precise version of this statement.

We say that a function of two variables $F(z, w)$ is holomorphic if it is holomorphic in each variable separately. If F is a holomorphic function of two variables we will denote the (complex) partial derivatives with respect to the first and second variables by F_1 and F_2 respectively. Then the following theorem is easy to prove if we assume that F is continuously differentiable (in the real sense); and this, actually, follows from the hypotheses on F as given, but that is not easy to show.

---

[9] This section is adapted from Appendix 7 "The Cauchy-Riemann Equations Revisited" in D.C. Ullrich: Complex Made Simple; AMS, Providence, RI (2008).

Theorem II.5.1. *Suppose that F is a holomorphic function of two variables and $f(z) = F(z, \overline{z})$. Then,*

$$\frac{\partial f}{\partial z}(z) \;=\; F_1(z, \overline{z}), \qquad \frac{\partial f}{\partial \overline{z}}(z) \;=\; F_2(z, \overline{z}),$$

and hence:

$$f \text{ is holomorphic iff } F_2 = 0, \text{ in which case } f'(z) = F_1(z, \overline{z}).$$

$\square$

E.g., we can interpret $f(z) = |z|^2$

- either as $f(z) = F(z, \overline{z}) = |z|^2$ (in which case from an exercise on p. 17 we know that $F(z, \overline{z})$ is holomorphic in the first variable only at $z = 0$)
- or as $f(z) = F(z, \overline{z}) = z\,\overline{z}$ (in which case F is holomorphic in both variables, but $F_2(z, \overline{z}) = z$).

III. Integration in the Complex Plane

III.1. Path Integrals

References: [DET, Sections 3.1 & 3.2] and [ST, Chapter 6]
(ML-inequality: [ST, Section 6.6]; FTC: [ST, Section 6.5])

One goal (not the only one): make sense of $\int_{z_0}^{z} f(w)\,\mathrm{d}w$, where possible.

The problem: how to get from z_0 to z?

The solution: use a path $\gamma : [a,b] \to \mathbb{C}$ with endpoints z_0 and z.

Definition. Let $f : D \to \mathbb{C}$ be continuous and $\gamma : [a,b] \to D$ a piecewise regular path. Then we define

$$\int_\gamma f\,\mathrm{d}z = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\ \mathrm{d}t = \int_a^b U(t)\ \mathrm{d}t + i\int_a^b V(t)\ \mathrm{d}t, \tag{III.1}$$

where $U(t) = \operatorname{Re}(f(\gamma(t)) \cdot \gamma'(t))$ and $V(t) = \operatorname{Im}(f(\gamma(t)) \cdot \gamma'(t))$ with $U, V : [a,b] \to \mathbb{R}$.

Remark. $U, V : [a,b] \to \mathbb{R}$ are piecewise continuous since f is continuous and γ is piecewise regular; hence the right-hand side integral exists.

Remark. If we use Leibniz notation, there is an almost irresistible temptation to write

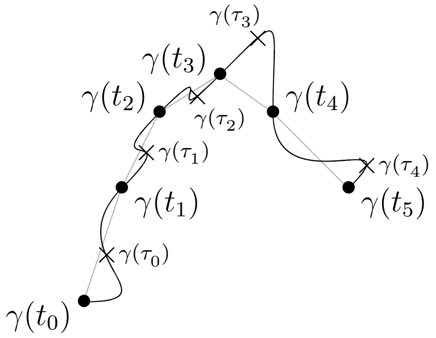
$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(z)\frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t.$$

That this notation is well chosen, can be justified by defining path integrals via Riemann sums (as in M11)

$$S_{\pi,\zeta}(f) = \sum_{k=1}^{n} f(\zeta_k)\ \mathrm{d}z_k,$$

where $\mathrm{d}z_k = \gamma(t_k) - \gamma(t_{k-1})$ and $\zeta_k = \gamma(\tau_k)$ with $t_{k-1} \le \tau_k \le t_k$, and define

$$\int_\gamma f\,\mathrm{d}z = \lim_{\max|\,\mathrm{d}z_k| \to 0} S_{\pi,\zeta}(f).$$



provided that this limit exists.
Then one proves: If $f : D \to \mathbb{C}$ is continuous on a domain D and $\gamma : [a,b] \to D$ is a piecewise regular path, then the integral is given as in (III.1). For details, see [DET, Sections 3.1 & 3.2] and [ST, Sections 6.1–6.3].

Comparison with the real line integral of a vector field. If $\vec{v} = (v_1, v_2) : D \to \mathbb{R}^2$ is a continuous vector field then (see M10)

$$\int_\gamma \vec{v} \cdot \mathrm{d}\vec{s} = \int_\gamma v_1 \ \mathrm{d}x + v_2 \ \mathrm{d}y = \int_a^b \vec{v}(\gamma(t)) \cdot \gamma'(t) \ \mathrm{d}t. \tag{III.2}$$

The difference is the multiplication:

- for $f : D \to \mathbb{C}$ the "$\cdot$" in $f(\gamma(t)) \cdot \gamma'(t)$ means *complex multiplication*, whereas
- for $\vec{v} : D \to \mathbb{R}^2$ the "$\cdot$" in $\vec{v}(\gamma(t)) \cdot \gamma'(t)$ means *scalar multiplication*, i.e., writing $\gamma = (\operatorname{Re}\gamma, \operatorname{Im}\gamma)$,

$$\vec{v}(\gamma(t)) \cdot \gamma'(t) = v_1(\gamma(t)) \ \operatorname{Re}\gamma'(t) + v_2(\gamma(t)) \ \operatorname{Im}\gamma'(t).$$

Example. If $\gamma : [a, b] \to \mathbb{C}$ is piecewise regular, then

$$\int_\gamma 1 \, \mathrm{d}z = \gamma(b) - \gamma(a)$$

(in contrast to the real line integral $\int_\gamma 1 \, \mathrm{d}x + 0 \, \mathrm{d}y = \operatorname{Re}\gamma(b) - \operatorname{Re}\gamma(a)$).

(ex) Verify that the complex path integral can be written in terms of two real path integrals

$$\int_\gamma f \, \mathrm{d}z = \int_\gamma u \ \mathrm{d}x - v \ \mathrm{d}y + i \int_\gamma v \ \mathrm{d}x + u \ \mathrm{d}y,$$

where $f = u + iv$.

(ex) Recall that for a continuous vector field $\vec{v} = (v_1, v_2) : D \to \mathbb{R}^2$ and a smooth path $\gamma : [a, b] \to D$, $\gamma(t) = (x(t), y(t))$ the line integral is defined by Eq. (III.2). One can show (we take it as a definition here!) that for a simple closed path γ

$$F(\gamma) = \int_\gamma x \ \mathrm{d}y$$

is the area enclosed by γ (i.e., the area of I_γ). How can we interpret the path integral $\int_\gamma \bar{z} \ \mathrm{d}z$ for a simple closed smooth path γ in $\mathbb{C}$?

Properties of the path integral.

(i) *Linearity of the integral:* The map $f \mapsto \int_\gamma f \, \mathrm{d}z$ is linear, i.e.,

$$\int_\gamma f + g \ \mathrm{d}z = \int_\gamma f \ \mathrm{d}z + \int_\gamma g \ \mathrm{d}z$$

and, for $c \in \mathbb{C}$,

$$\int_\gamma cf \ \mathrm{d}z = c \int_\gamma f \ \mathrm{d}z.$$

Proof. ...directly from the propositions of real integrals (ex). □

(ii) *Path-additivity of the integral:* If $\gamma : [a,b] \to D$ and $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to D$ with $\gamma(b) = \tilde{\gamma}(\tilde{a})$ then

$$\int_{\gamma+\tilde{\gamma}} f \,\mathrm{d}z = \int_{\gamma} f \,\mathrm{d}z + \int_{\tilde{\gamma}} f \,\mathrm{d}z,$$

and, for the inverse path $(-\gamma)$,

$$\int_{-\gamma} f \,\mathrm{d}z = -\int_{\gamma} f \,\mathrm{d}z.$$

Proof. …directly from the propositions of real integrals (ex). ☐

(iii) *Parameter invariance:* If $\gamma : [a,b] \to D$ and $\tilde{\gamma} = \gamma \circ h : [\tilde{a}, \tilde{b}] \to D$ are two parametrizations of a contour $\mathit{\Gamma} = \gamma([a,b])$ then

$$\int_{\tilde{\gamma}} f \,\mathrm{d}z = \begin{cases} +\int_{\gamma} f \,\mathrm{d}z & \text{if } h' > 0, \\ -\int_{\gamma} f \,\mathrm{d}z & \text{if } h' < 0. \end{cases}$$

Proof. As $h : [\tilde{a}, \tilde{b}] \to [a,b]$ is regular, h' does not change sign and $h(\tilde{a}) = a$, $h(\tilde{b}) = b$ if $h' > 0$ and $h(\tilde{a}) = b$, $h(\tilde{b}) = a$ if $h' < 0$. Now compute

$$\int_{\tilde{\gamma}} f \,\mathrm{d}z = \int_{\tilde{a}}^{\tilde{b}} \left((f \circ \gamma) \cdot \gamma'\right)(h(t)) \cdot h'(t) \,\mathrm{d}t = \pm \int_{a}^{b} \left((f \circ \gamma) \cdot \gamma'\right)(t) \,\mathrm{d}t,$$

depending on whether $h' > 0$ or $h' < 0$. ☐

(ex) Compute the path integrals $\int_{\gamma_i} f_j \,\mathrm{d}z$ for

$$f_1(z) = z, \qquad f_2(z) = \overline{z}, \qquad \text{and} \qquad f_3(z) = |z|^2,$$

where

- γ_1 is the straight line segment from $z = 0$ to $z = 1 + i$, and
- γ_2 is the polygonal path from $z = 0$ to $z = 1 + i$ via $z = 1$.

What do you observe?

Orientation of a contour. If $\mathit{\Gamma} \subset \mathbb{C}$ is a simple contour then the integral $\int_{\gamma} f \,\mathrm{d}z$, where γ is a parametrization of $\mathit{\Gamma}$, is determined up to sign:

- Any two regular parametrizations of a (regular) piece of $\mathit{\Gamma}$ are related by $\tilde{\gamma} = \gamma \circ h$ so that the integral is unique up to sign, by (iii), and the sign is picked by the choice of initial point.
- The first choice of initial point determines the initial point of any subsequent regular piece (after a partition has been fixed).
- The final integral does not depend on possibly different partitions into pieces, by (ii).

The same arguments justify the assertion for simple closed contours.

The assertion is not true for non-simple contours: for example, for each loop of a figure-∞ contour there are two possible orientations so that, in general, there are four different values for the integral.

We refer to the sense of parametrization of a simple (closed) contour as its *orientation*; for a simple closed contour we will (unless stated otherwise) take the *counter-clockwise* orientation when writing the *contour integral*

$$\int_\Gamma f \,\mathrm{d}z.$$

An alternative approach to orientation uses equivalence classes:

ex We call two regular paths $\gamma : [a,b] \to \mathbb{C}$ and $\tilde{\gamma} : [\tilde{a},\tilde{b}] \to \mathbb{C}$ *equivalent*, $\tilde{\gamma} \sim \gamma$, if there is a regular and onto $h : [\tilde{a},\tilde{b}] \to [a,b]$ with $\tilde{\gamma} = \gamma \circ h$ so that $h' > 0$. Prove that $\sim$ is an equivalence relation.

Much of complex analysis depends on the fact that the following integral for $k = -1$ does not vanish.

Example.

$$\int_{|z-z_0|=r} (z-z_0)^k \,\mathrm{d}z = \begin{cases} 2\pi i & \text{if } k = -1, \\ 0 & \text{otherwise,} \end{cases}$$

for $k \in \mathbb{Z}$.

Note: If one wants to emphasise that one integrates around a *closed* contour, the notations $\oint_{|z-z_0|=r} (z-z_0)^k \,\mathrm{d}z$ and $\oint_{\partial B_r(z_0)} (z-z_0)^k \,\mathrm{d}z$ are used.

Proof. We parametrize the circle $\partial B_r(z_0)$ by $[0,2\pi] \ni t \mapsto \gamma(t) = z_0 + r\,\mathrm{e}^{it}$. Note that this is a counterclockwise parametrization of the circle.

Thus we compute:

$$\begin{aligned} \int_{\partial B_r(z_0)} (z-z_0)^k \,\mathrm{d}z &= \int_0^{2\pi} (\gamma(t)-z_0)^k \gamma'(t) \,\mathrm{d}t = \int_0^{2\pi} r\,\mathrm{e}^{k\,it}\; ir\,\mathrm{e}^{it} \,\mathrm{d}t \\ &= ir^{k+1} \int_0^{2\pi} \cos((k+1)t) + i\sin((k+1)t) \,\mathrm{d}t \\ &= \begin{cases} 2\pi i & \text{if } k = -1, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

since 2π is a period for $\cos(jt)$ and $\sin(jt)$ for any $j \neq 0$. □

Lemma III.1.1 (ML-inequality)**.** *Let $f : \gamma([a,b]) \to \mathbb{C}$ be continuous (where $\gamma([a,b])$ is a contour) and denote $M = \max_{z\in\gamma([a,b])} |f(z)|$, and let L be the length of $\gamma([a,b])$; then*

$$\left| \int_\gamma f \,\mathrm{d}z \right| \leq ML.$$

Remark. Note that $|f \circ \gamma| : [a,b] \longrightarrow \mathbb{R}$ is continuous and hence

$$M = \max_{z \in \gamma([a,b])} |f(z)| = \max_{t \in [a,b]} |f \circ \gamma(t)|$$

exists by Weierstrass' Theorem (in $\mathbb{R}$).

Proof. It is enough to prove[1] this for a regular path γ (instead of a piecewise regular path). Let

$$g(t) = f(\gamma(t)) \cdot \gamma'(t) = U(t) + iV(t)$$

and let $\int_\gamma f(z)\,\mathrm{d}z = r\,\mathrm{e}^{i\varphi} \in \mathbb{C}$. Then, we have (using only real analysis)

$$\begin{aligned} \left| \int_\gamma f(z)\,\mathrm{d}z \right| &= \left| \int_a^b g(t)\,\mathrm{d}t \right| = r = \operatorname{Re} r = \operatorname{Re}\left(\mathrm{e}^{-i\varphi} \int_a^b g(t)\,\mathrm{d}t \right) \\ &= \int_a^b \operatorname{Re}\left(e^{-i\varphi} g(t) \right) \mathrm{d}t \leq \int_a^b \left| e^{-i\varphi} g(t) \right| \mathrm{d}t = \int_a^b |g(t)|\,\mathrm{d}t \\ &= \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\,\mathrm{d}t \leq \int_a^b M \cdot |\gamma'(t)|\,\mathrm{d}t = M \cdot L. \end{aligned}$$

□

Remark. This result is not hard to prove[2] but it is of immense theoretical importance: Most of the major results of complex analysis involve at least one use of it.

(ex) Let $\Gamma \subset \mathbb{C}$ be a simple closed contour and $z_0 \in I_\Gamma$ a point in its interior. Prove that there is a $\varrho > 0$ so that

$$\left| \int_\Gamma \frac{\mathrm{d}z}{(z - z_0)^k} \right| < \frac{1}{\varrho^k} L,$$

where $k \in \mathbb{N}$ and L is the length of Γ.

---

[1] Alternative proofs can be found in [ST, Section 6.6] (using a slighly different method to obtain integrals over real functions only) or, using Riemann sums, in [DET, pp. 85–86] and [ST, Section 6.6]. We give the proof of [ST, Section 6.6] here:
First we show $|\int_a^b U(t) + iV(t)\,\mathrm{d}t| \leq \int_a^b |U(t) + iV(t)|\,\mathrm{d}t$: thus we write $X + iY = \int_a^b U(t) + iV(t)\,\mathrm{d}t$; then

$$\begin{aligned} X^2 + Y^2 &= \int_a^b (XU(t) + YV(t))\,\mathrm{d}t \\ &\leq \int_a^b \sqrt{X^2 + Y^2}\,\sqrt{U^2(t) + V^2(t)}\,\mathrm{d}t \\ &= \sqrt{X^2 + Y^2} \int_a^b \sqrt{U^2(t) + V^2(t)}\,\mathrm{d}t. \end{aligned}$$

Now we find

$$\left| \int_\gamma f\,\mathrm{d}z \right| \leq \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,\mathrm{d}t \leq \int_a^b M|\gamma'(t)|\,\mathrm{d}t = ML$$

from real analysis.

[2] Also, compare this estimate with the corresponding estimate in real analysis $\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x$.
For this, we look at the following example:
Let $f(z) = 1/z$ and $\gamma : [0, 2\pi] \ni t \mapsto \mathrm{e}^{i\,t}$. Then $|f(\gamma(t))| = 1$ and $|\gamma'(t)| = 1$ (and $L = 2\pi$). The ML-estimate is $\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq 2\pi$.
However, observe that $\int_\gamma |f(z)|\,\mathrm{d}z = \int_\gamma 1\,\mathrm{d}z = 0$; so the ML-estimate is *not* the above estimate from real analysis with z replacing x naively (compare, however, to the previous footnote)!

ⓔⓧ Let $\gamma : [a,b] \to \mathbb{C}$ be a closed regular path. For $z \notin \gamma([a,b])$ we define the *winding number* of γ around z by

$$w(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z}.$$

Prove that

(i) $w(\gamma, z) \in \mathbb{Z}$
Hint: write $\varphi(t) = \int_a^t \frac{\gamma'(\tau)\,\mathrm{d}\tau}{\gamma(\tau)-z}$ and show that $(\gamma(t) - z)\,\mathrm{e}^{-\varphi(t)}$ is constant – remember that $\mathrm{e}^{x+iy} = \mathrm{e}^x(\cos y + i \sin y)$.

(ii) $z \mapsto w(\gamma, z)$ is constant on each connected set $A \subset \mathbb{C} \setminus \gamma([a,b])$
Hint: first prove that $z \mapsto w(\gamma, z)$ is continuous by using the ML-inequality.

Lemma III.1.2 (Fundamental theorem of calculus for path integrals)**.** *Let $f : D \to \mathbb{C}$ be continuous and suppose f has an anti-derivative $F : D \to \mathbb{C}$, i.e., F is holomorphic on D with $F' = f$. Then, the following hold and are equivalent:*

(i) *Let $\gamma : [a,b] \to D$ be a piecewise regular path. Then $\int_\gamma f\,\mathrm{d}z$ only depends on the endpoints $\gamma(a)$ and $\gamma(b)$ of the path γ; more precisely, $\int_\gamma f\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$.*

(ii) *$\int_\Gamma f\,\mathrm{d}z = 0$ for every closed contour $\Gamma \subset D$.*

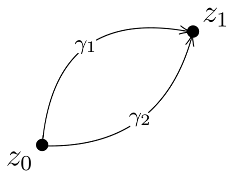
Proof. (i): For a regular path $\gamma : [a,b] \to D$ we have

$$\int_\gamma f\,\mathrm{d}z = \int_a^b F'(\gamma(t))\,\gamma'(t)\,\mathrm{d}t = \int_a^b (F \circ \gamma)'(t)\,\mathrm{d}t = F(\gamma(b)) - F(\gamma(a))$$

by the chain rule. For a piecewise regular path $\gamma_1 + \cdots + \gamma_n : [a,b] \to D$ with regular $\gamma_k : [t_{k-1}, t_k] \to D$, where $t_0 = a$ and $t_n = b$, we therefore find

$$\int_{\gamma_1 + \cdots + \gamma_n} f\,\mathrm{d}z = \sum_{k=1}^n F(\gamma(t_k)) - F(\gamma(t_{k-1})) = F(\gamma(b)) - F(\gamma(a)).$$

(ii): follows trivially from (i) since $\gamma(b) = \gamma(a)$.
(ii)⇒(i): We have to show that the path integral only depends on the endpoints of the path. If γ_1 and γ_2 are two piecewise regular paths joining two points z_0, z_1 then $\gamma_1 - \gamma_2$ is a piecewise regular closed path so that



$$0 = \int_{\gamma_1 - \gamma_2} f\,\mathrm{d}z = \int_{\gamma_1} f\,\mathrm{d}z - \int_{\gamma_2} f\,\mathrm{d}z.$$

Consequently, the path integral is independent of the actual path (joining z_0 and z_1) chosen. ☐

ⓔⓧ Prove that the function $f(z) = \frac{1}{z}$ does not have an anti-derivative in any punctured neighbourhood of the origin $z = 0$ (e.g., $B_\varepsilon(0) \setminus \{0\}$).

Question: Under which conditions does f have an anti-derivative?

III.2. Cauchy's Theorem

References: [DET, Sections 3.3 & 3.4] and [ST, Chapters 8 & 9]
(Cauchy's Theorem: [DET, Theorem 3.3.1] and [ST, Section 8.5]; Cauchy's Theorem for simply connected domains: [DET, Theorem 3.4.3] and [ST, Theorem 8.8]; Cauchy-Goursat Theorem: [DET, Lemma 3.3.1] and [ST, Theorem 8.1]; Homotopy version of Cauchy's Theorem: [DET, Theorem 3.4.5] and [ST, Theorem 9.4])

The following lemma is the converse to Lemma III.1.2 and provides the ultimate answer to our initial question: Under which conditions does f have an anti-derivative, i.e., when does it make sense to write $\int_{z_0}^{z} f(z)\,\mathrm{d}z$?".

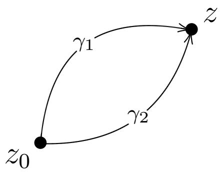
Lemma III.2.1 (Criterion for the FTC)**.** *If $f : D \to \mathbb{C}$ is continuous on a domain D and $\int_\Gamma f\,\mathrm{d}z = 0$ for every closed contour $\Gamma \subset D$ then f has an anti-derivative $F : D \to \mathbb{C}$.*

Proof. Fix $z_0 \in D$ and define

$$F(z) = \int_\gamma f\,\mathrm{d}z,$$

where γ is *any* piecewise regular path joining z_0 to z.

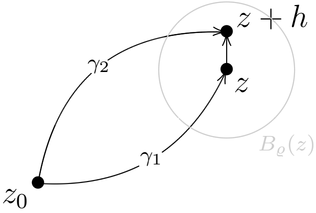
$F : D \to \mathbb{C}$ is well defined since $\int_\Gamma f\,\mathrm{d}z = 0$ for every closed contour $\Gamma \subset D$: if γ_1 and γ_2 are two piecewise regular paths joining z_0 to z then $\gamma_1 - \gamma_2$ is a piecewise regular closed path so that (also see above)



$$\int_{\gamma_1} f\,\mathrm{d}z = \int_{\gamma_2} f\,\mathrm{d}z.$$

To see that F is holomorphic with $F' = f$, fix $z \in D$ and take $\varrho > 0$ so that $B_\varrho(z) \subset D$. Then, for $|h| < \varrho$, let γ_1 be a path joining z_0 and z and γ_2 a path joining z_0 and $z+h$ and calculate

$$\begin{aligned} |F(z+h) - F(z) - f(z)h| &= \left| \int_{\gamma_2} f\,\mathrm{d}z - \int_{\gamma_1} f\,\mathrm{d}z - f(z)h \right| \\ &\overset{\star}{=} \left| \int_0^1 f(z+th)h\,\mathrm{d}t - f(z)h \right| \\ &= \left| \int_0^1 (f(z+th) - f(z))h\,\mathrm{d}t \right| \\ &\overset{ML\text{-ineq.}}{\leq} \max_{t\in[0,1]} |f(z+th) - f(z)|\,|h|, \end{aligned}$$



where in the step "$\star$" we used that $\int_\Gamma f\,\mathrm{d}z = 0$ for any closed contour.

Thus $\frac{F(z+h)-F(z)}{h} \to f(z)$ as $|h| \to 0$ since f is continuous at z. ☐

The following is the main theorem of this course(!), it puts forward conditions under which $\int_\Gamma f\,\mathrm{d}z = 0$ when there is no initial reason for f to have an anti-derivative.

Theorem III.2.2 (Cauchy's Theorem)**.** *Let $f : D \to \mathbb{C}$ be holomorphic in a domain D and let $\Gamma \subset D$ be a simple closed contour so that its interior $I_\Gamma \subset D$. Then $\int_\Gamma f\,\mathrm{d}z = 0$.*

Corollary III.2.3 (Cauchy's Theorem for simply connected domains)**.** *Let $f : D \to \mathbb{C}$ be holomorphic in a simply connected domain D. Then*

$$\int_\Gamma f \,\mathrm{d}z = 0$$

for any (simple) closed contour $\Gamma \subset D$.

Remark. Cauchy's Theorem directly implies the Corollary for *simple* closed contours Γ but it can be proven to hold for *any* closed contour Γ, see for example [DET, Proof of Theorem 3.4.3].

The previous Corollary III.2.3 together with the Criterion for the FTC for path integrals (Lemma III.2.1) yields the following corollary.

Corollary III.2.4 (Existence of anti-derivatives)**.** *If $f : D \to \mathbb{C}$ is holomorphic in a simply connected domain D, then it has an anti-derivative $F : D \to \mathbb{C}$.* ☐

There are various versions[3] of Cauchy's Theorem. Our proofs, however, will only work for simple closed contours Γ, or for rather special domains D, respectively.

We shall prove the theorem in two special versions:

(i) when additionally f' is continuous in D, or

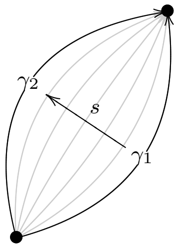
(ii) when Γ is a triangle.

Proof of Cauchy's Theorem when f' is continuous in D.

Definition. Two smooth paths $\gamma_1, \gamma_2 : [0,1] \to D \subset \mathbb{C}$ with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$ can be *smoothly deformed into each other* within D with their endpoints fixed, if there exists a twice continuously differentiable function[4] $C : [0,1] \times [0,1] \to D$ such that for all $t \in [0,1]$

$$\gamma_1(t) = C(0,t) \qquad \text{and} \qquad \gamma_2(t) = C(1,t)$$

and, for all $s \in [0,1]$, $C(s,0) = \gamma_i(0)$ and $C(s,1) = \gamma_i(1)$ remain constant.



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[3] For example, one can relax the holomorphicity assumption: it suffices to require f to be holomorphic on I_Γ and continuous on $\bar{I}_\Gamma = I_\Gamma \cup \Gamma$ (see [DET, Theorem 3.4.2]).

[4] The function C is a special ("smooth") case of a homotopy. In fact, a *continuous* function $\tilde{C} : [0,1] \times [0,1] \to D$ with the above properties is a *(fixed-endpoint) homotopy*, see [ST, Section 9.4]. In general, a *homotopy* in a domain D between two paths $\gamma_1 : [a,b] \to D$ and $\gamma_2 : [a,b] \to D$ is a continuous map $\tilde{C} : [0,1] \times [a,b] \to D$ such that for all $t \in [a,b]$

$$\gamma_1(t) = \tilde{C}(0,t) \qquad \text{and} \qquad \gamma_2(t) = \tilde{C}(1,t)$$

Theorem III.2.5 (Cauchy's Theorem (weak version I))**.** *Let $f : D \to \mathbb{C}$ be holomorphic in a simply connected domain D, with continuous f', and let $\gamma_1, \gamma_2 : [0,1] \to D$ be two smooth paths which have common endpoints and can be smoothly deformed into each other within D. Then*

$$\int_{\gamma_1} f \, \mathrm{d}z = \int_{\gamma_2} f \, \mathrm{d}z. \tag{III.3}$$

Proof. Set

$$I(s) = \int_{C(s,t)} f(z) \, \mathrm{d}z = \int_0^1 f(C(s,t)) \frac{\partial C(s,t)}{\partial t} \, \mathrm{d}t.$$

Then Eq. (III.3) is equivalent to $I(0) = I(1)$, which will be proved if we can show that $I'(s) = 0$ for $0 < s < 1$. Reversing – using Lemma III.2.6 – the order of differentiation and integration, we obtain

$$\begin{aligned} I'(s) = \frac{\mathrm{d}}{\mathrm{d}s} I(s) &= \int_0^1 \frac{\partial}{\partial s} \left(f(C(s,t)) \frac{\partial C(s,t)}{\partial t} \right) \mathrm{d}t \\ &= \int_0^1 \left(f'(C(s,t)) \frac{\partial C}{\partial s} \frac{\partial C}{\partial t} + f(C(s,t)) \frac{\partial^2 C}{\partial s \partial t} \right) \mathrm{d}t \\ &= \int_0^1 \frac{\partial}{\partial t} \left(f(C(s,t)) \frac{\partial C(s,t)}{\partial s} \right) \mathrm{d}t \\ &= f(C(s,t)) \frac{\partial C(s,t)}{\partial s} \bigg|_{t=0}^{t=1}, \end{aligned}$$

which is zero because $C(s,1)$ and $C(s,0)$ are constants (thus $\frac{\partial C(s,0)}{\partial s} = 0 = \frac{\partial C(s,1)}{\partial s}$). □

Remark. The assumption that $C(s,t)$ is twice continuously differentiable is needlessly strong. We need to require[5] only that C_s, C_t, C_{st} and C_{ts} are continuous and that $C_{st} = C_{ts}$.

Also note if γ_1 is closed, taking the path $\gamma_2(t)$ to be the single point $\gamma_1(0)$, yields the closed contour form of Cauchy's theorem: $\int_\Gamma f \, \mathrm{d}z = 0$, where γ_1 is the parametrisation of the closed contour Γ.

The following lemma was used to reverse the order of differentiation and integration.

Lemma III.2.6. *Let $f(z,w)$ and*[6] *$f_z(z,w)$ be continuous for z in a domain D and w on a simple contour Γ. Then $F(z) = \int_\Gamma f(z,w) \, \mathrm{d}w$ is holomorphic in D, and*

$$F'(z) = \int_\Gamma f_z(z,w) \, \mathrm{d}w.$$

ex Fill in the details (all "__________") in the following proof of Lemma III.2.6.

---

[5] Even these conditions need only hold piecewise, in the sense that there exists a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ such that the conditions above hold on each strip $[0,1] \times [t_{k-1}, t_k]$.

[6] So, $f(z,w)$ is complex differentiable with respect to z, i.e.,

$$f_z(z_0, w) = \lim_{z \to z_0} \frac{f(z,w) - f(z_0,w)}{z - z_0}.$$

Proof. Let $F(z) = U(x, y) + iV(x, y)$ and $f(z, w) = u\,(x, y, \xi(t), \eta(t)) + iv\,(x, y, \xi(t), \eta(t))$, where $\gamma(t) = \xi(t) + i\eta(t)$ is a parametrisation of Γ with $t \in [a, b]$. Then

$$U(x, y) = \int_a^b \underline{\qquad\qquad} \qquad \text{and} \qquad V(x, y) = \int_a^b \underline{\qquad\qquad}.$$

If $z_0 = x_0 + iy_0 \in D$, then (here $h \in \mathbb{R}$)

$$\begin{aligned} U_x(x_0, y_0) = \lim_{h\to 0} \frac{U(x_0 + h, y_0) - U(x_0, y_0)}{h} \quad &= \quad \lim_{h\to 0} \frac{1}{h} \int_a^b \underline{\qquad\qquad} \\ \overset{\text{MVT}}{=} \lim_{h\to 0} \int_a^b \underline{\qquad\qquad} \quad &= \quad \int_a^b [u_x(x_0, y_0, \xi, \eta)\, \xi' - v_x(x_0, y_0, \xi, \eta)\, \eta']\, \mathrm{d}t. \end{aligned}$$

The justification for taking the limit under the integral sign in the last step is supplied by __________ and the following lemma (for a proof see [DET, Theorem 3.5.2]):

> **Lemma.** Let $f(z, w)$ be continuous for z in a domain D and w on a simple contour Γ. Then $F(z) = \int_\Gamma f(z, w)\, \mathrm{d}w$ is continuous in D. ☐

Similarly,

$$U_y(x_0, y_0) = \int_a^b \underline{\qquad\qquad}, \qquad\qquad V_x(x_0, y_0) = \int_a^b \underline{\qquad\qquad},$$

$$V_y(x_0, y_0) = \int_a^b \underline{\qquad\qquad}.$$

Since $f(z, w)$ is holomorphic at z_0, we have $u_x(x_0, y_0, \xi, \eta) = v_y(x_0, y_0, \xi, \eta)$ and __________=__________, and therefore

$$U_x(x_0, y_0) = \underline{\qquad\qquad} \qquad \text{and} \qquad \underline{\qquad\qquad} = \underline{\qquad\qquad}.$$

Hence, $F(z)$ is holomorphic at z_0 and

$$F'(z_0) = U_x(x_0, y_0) + i\, V_x(x_0, y_0) = \int_a^b \underline{\qquad\qquad} = \int_\Gamma \underline{\qquad\qquad}.$$

Since z_0 is any point in D, the result holds throughout D. ☐

Alternatively, one can also prove a weak version using Green's Theorem (cf. M10 or [DET, Theorem 3.1.2]).

Theorem (Green's Theorem)**.** *Let $\alpha, \beta : \mathbb{R}^2 \supset D \to \mathbb{R}$ be continuously differentiable and $\Omega \subset D$ bounded with piecewise smooth boundary $\partial\Omega$. Then*

$$\int_{\partial\Omega} \alpha\ \mathrm{d}x + \beta\ \mathrm{d}y = \iint_\Omega (\beta_x - \alpha_y)\ \mathrm{d}x\, \mathrm{d}y,$$

where the boundary integral is taken in the counter-clockwise sense. ☐

Theorem III.2.7 (Cauchy's Theorem (weak version II))**.** *Let $f : D \to \mathbb{C}$ be holomorphic in a domain D, with continuous f', and let $\Gamma \subset D$ be a simple closed contour so that its interior $I_\Gamma \subset D$. Then $\int_\Gamma f\, \mathrm{d}z = 0$.*

(ex) Let $f : D \to \mathbb{C}$ be holomorphic in a domain D, with continuous f', and let $\Gamma \subset D$ be a simple closed contour so that its interior $I_\Gamma \subset D$. Use Green's theorem to show that $\int_\Gamma f \, \mathrm{d}z = 0$.

Both weak versions can be used to establish:

Corollary III.2.3' (Cauchy's Theorem for simply connected domains)**.** *Let $f : D \to \mathbb{C}$ be holomorphic in a simply connected domain D (with continuous derivative f'). Then*

$$\int_\Gamma f \, \mathrm{d}z = 0$$

for any (simple) closed contour $\Gamma \subset D$. □

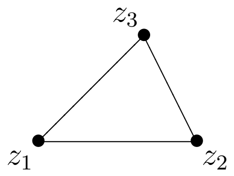
Remark. (i) Our proofs require continuity of f' but the more general statement of Cauchy's Theorem III.2.2 above yields the corollary without that assumption (cf. [DET, Theorem 3.3.1]).

(ii) The proof using smooth deformations requires C_s, C_t etc. to be piecewise continuous, but it can be formulated and holds with C continuous (cf. [ST, Theorem 9.3]).

(iii) The proof using Green's Theorem requires Γ to be *simple* closed but (in contrast to the above version of Cauchy's Theorem III.2.2) it can be formulated and holds without it (cf. [DET, Proof of Theorem 3.4.3]).

Proof of Cauchy's Theorem when Γ is a triangle.

Definition. A *triangle* is a simple closed polygonal contour consisting of three line segments, $\Delta = [z_2, z_3] \cup [z_3, z_1] \cup [z_1, z_2]$ (where $[z, \tilde{z}] = \{z + t\,(\tilde{z} - z) | t \in [0, 1]\}$).



Theorem III.2.8 (Cauchy-Goursat Theorem)**.** *Let $f : D \to \mathbb{C}$ be holomorphic in a domain D and let $\Delta \subset D$ be a triangle so that its interior $I_\Delta \subset D$. Then $\int_\Delta f \, \mathrm{d}z = 0$.*

Proof. We first give an outline of the steps in this proof:

(i) Construct a sequence of triangles (Δ_n) with $\Delta_1 = \Delta$ such that $\left|\int_{\Delta_n} f \, \mathrm{d}z\right| \le 4 \left|\int_{\Delta_{n+1}} f \, \mathrm{d}z\right|$ for all n.

(ii) Obtain the length of Δ_n.

(iii) Show that there is a point z_0 belonging to all triangles Δ_n.

(iv) Show that $|z_0 - z| \le \left(\frac{1}{2}\right)^n L$ where L is the length of Δ.

(v) Show that $f(z)$ may be expressed on (Δ_n) as

$$f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\eta(z),$$

where $\lim_{z \to z_0} \eta(z) = 0$.

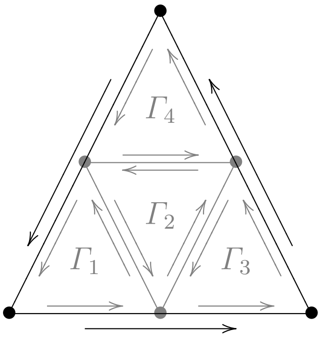
(vi) Obtain an estimate for $\left|\int_{\Delta_n} f \, \mathrm{d}z\right|$ by using the ML-inequality.

For the proof, let L denote the length of (the perimeter of) Δ and $I = |\int_\Delta f \, \mathrm{d}z|$.

(i) We construct a sequence of triangles Δ_n with

$$I \leq 4^{n-1} I_n = 4^{n-1} \left| \int_{\Delta_n} f \, \mathrm{d}z \right|. \tag{III.4}$$

Let $\Delta_1 = \Delta$, and suppose that Δ_n with (III.4) has just been constructed. Join the midpoints of the sides of Δ_n to produce four similar triangles Γ_j, $j = 1, 2, 3, 4$. Then, by the path-additivity of the integral



$$I_n = \left| \sum_{j=1}^{4} \int_{\Gamma_j} f \, \mathrm{d}z \right| \leq \sum_{j=1}^{4} \left| \int_{\Gamma_j} f \, \mathrm{d}z \right|.$$

Let k be such that $\left| \int_{\Gamma_k} f \, \mathrm{d}z \right|$ is the largest of $\left| \int_{\Gamma_1} f \, \mathrm{d}z \right|, \dots, \left| \int_{\Gamma_4} f \, \mathrm{d}z \right|$ and set $\Delta_{n+1} = \Gamma_k$. Then

$$I \leq 4^{n-1} I_n \leq 4^{n-1} \cdot 4 I_{n+1} = 4^n I_{n+1}.$$

So we have obtained a sequence $(\Delta_n)_{n \in \mathbb{N}}$ by induction.

Note: It now remains to show that $\lim_{n \to \infty} 4^{n-1} I_n = 0$.

(ii) The length L_n of the triangle Δ_n: By the construction of Δ_{n+1}, the length of each side of Δ_{n+1} is half the length of a side in Δ_n, thus $L_{n+1} = \frac{1}{2} L_n$ and so (by induction) $L_n = \left(\frac{1}{2}\right)^{n-1} L$.

Note: Considering $I \leq 4^{n-1} \left| \int_{\Delta_n} f \, \mathrm{d}z \right|$, we thus need an upper bound for $|f|$ on Δ_n (the M in the ML-inequality) which is better than $\left(\frac{1}{2}\right)^{n-1}$.

(iii) The sequence of triangles Δ_n has a "limit point" z_0, i.e.,

$$\forall \delta > 0 \, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N : \Delta_n \subset B_\delta(z_0).$$

Denote by $T_n = \Delta_n \cup I_{\Delta_n}$ the triangle Δ_n with its interior I_{Δ_n} (so, T_n is the filled triangle). All T_n are compact, $T_{n+1} \subset T_n$, and $\operatorname{diam} T_n \leq \frac{1}{2} L_n \to 0$ as $n \to \infty$. By Cantor's Intersection Theorem (see p. 10) there is a point $z_0 \in \mathbb{C}$ so that

$$\{z_0\} = \bigcap_{n \in \mathbb{N}} T_n.$$

(iv) For any $z \in T_n$, we have

$$\forall z \in T_n : |z - z_0| \leq \operatorname{diam} T_n \leq \frac{1}{2} L_n.$$

(v) We use that f is differentiable at $z = z_0$: writing

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0)$$

differentiablity means that

$$\forall \varepsilon > 0 \, \exists \delta > 0 : |z - z_0| < \delta \Rightarrow |\eta(z)| < \varepsilon,$$

that is, η is continuous on $\mathbb{C}$ when setting $\eta(z_0) = 0$.

(vi) The integral nearly vanishes on small triangles: we show

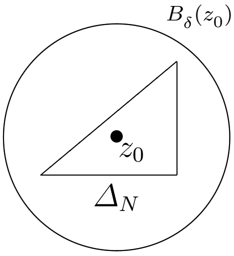
$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N : I_n \leq \varepsilon \frac{L_n^2}{2}.$$

Writing

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0),$$

we note that $f(z_0) + f'(z_0)(z - z_0)$ is the derivative of $z \mapsto z\, f(z_0) + f'(z_0)\frac{(z-z_0)^2}{2}$. So, by the FTC for path integrals, the integral $\int_{\Delta_n} (f(z_0) + f'(z_0)(z - z_0))\, \mathrm{d}z$ vanishes. Now fix $\varepsilon > 0$, $\delta > 0$ as above and let $N \in \mathbb{N}$ such that $\Delta_N \subset B_\delta(z_0)$. Then

$$\begin{aligned} I_n &= \left| \int_{\Delta_n} f(z)\, \mathrm{d}z \right| = \left| 0 + \int_{\Delta_n} \eta(z)(z - z_0)\, \mathrm{d}z \right| \\ &\leq L_n \cdot (\varepsilon \cdot \frac{1}{2} L_n) = \frac{\varepsilon\, L^2}{2 \cdot 4^{n-1}} \end{aligned}$$



for all $n \geq N$ by the ML-inequality.

Conclusion: Fix $\varepsilon > 0$ and let $N \in \mathbb{N}$ be as in (vi); then, for $n \geq N$,

$$I \leq 4^{n-1} I_n \leq 4^{n-1} \frac{\varepsilon\, L^2}{2 \cdot 4^{n-1}} = \varepsilon \frac{L^2}{2},$$

and we find $I = 0$, as desired, since $\varepsilon > 0$ was arbitrary. □

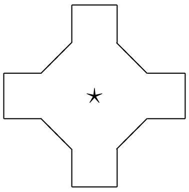
Remark. This constitutes the first part of a proof for the above more general version of Cauchy's Theorem. The next step would be to show it for (convex) polygons, then arbitrary closed polygonal contours (both by triangulation), and finally using that arbitrary simple closed contours can be well-approximated by a polygonal contour; compare [DET, Section 3.3 & Proof of Theorem 3.4.3] and [ST, Section 8.5]. The result will then be Theorem III.2.2.

Applications and Generalizations.

Definition. A domain $D \subset \mathbb{C}$ is called a *star-domain* if

$$\exists z_0 \in D \, \forall z \in D : [z_0, z] = \{(1 - t)z_0 + tz \,|\, t \in [0, 1]\} \subset D.$$

We call z_0 a *centre* of the star-domain.

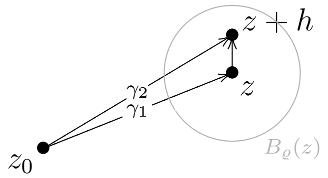


Example. Any disk $B_r(z)$ is a star-domain and, more generally, any open *convex set* C is a star-domain with any $z_0 \in C$ as a centre ($C \subset \mathbb{C}$ is convex if, for any pair of points $z, \tilde{z} \in C$, the line segment $[z, \tilde{z}] \subset C$).

We can now prove the existence of anti-derivatives on star-domains from the Cauchy-Goursat theorem (without using the general version of Cauchy's theorem) by replacing the general paths in the above proof by line segments:

Corollary III.2.9 (Existence of anti-derivatives)**.** *If $f : D \to \mathbb{C}$ is holomorphic in a star-domain D then it has an anti-derivative $F : D \to \mathbb{C}$.*

Proof. Let z_0 be the centre of the star-domain and define $F(z) = \int_{[z_0,z]} f \, \mathrm{d}z$. Then imitate the proof of the criterion for the FTC (Lemma III.2.1) and use the Cauchy-Goursat Theorem (ex). ☐



Now, the Cauchy-Goursat theorem yields Cauchy's theorem on star-domains:

Corollary III.2.10 (Cauchy's Theorem for star-domains)**.** *Let $f : D \to \mathbb{C}$ be holomorphic in a star-domain D. Then*

$$\int_\Gamma f \, \mathrm{d}z = 0$$

for any closed contour $\Gamma \subset D$.

Proof. This follows from the existence of an anti-derivative and the FTC for path integrals (Lemma III.1.2). ☐

Remark. Note that we do not need to assume Γ to be *simple* closed.

(ex) Prove that the function $f(z) = \frac{1}{z}$ has an anti-derivative F in the *cut plane* $\mathbb{C} \setminus \mathbb{R}_{\leq 0} = \{z = r\,\mathrm{e}^{i\varphi} \,|\, r > 0, -\pi < \varphi < \pi\}$. Set $F(1) = 0$, what do you get for $F(z)$ with $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$?

Hint: For the second part, find a suitable path joining 1 and $z = r\,\mathrm{e}^{i\varphi}$.

Remark: This anti-derivative with $F(1) = 0$ is called the *principal value of the (complex) logarithm* and denoted $\mathrm{Log}(z)$.

And here comes an important generalization[7] of Cauchy's theorem:

Theorem III.2.11 (Cauchy's Theorem (Homotopy version))**.** *Let $f : D \to \mathbb{C}$ be holomorphic and $\Gamma_1 \subset D$ and $\Gamma_2 \subset I_{\Gamma_1}$ simple closed contours such that $(I_{\Gamma_1} \setminus I_{\Gamma_2}) \subset D$. Then*

$$\int_{\Gamma_1} f \, \mathrm{d}z = \int_{\Gamma_2} f \, \mathrm{d}z.$$

Proof. We take two (piecewise) regular simple and disjoint paths that "join" Γ_1 and Γ_2,

$$\alpha_1, \alpha_2 : [1,2] \to (\Gamma_1 \cup I_{\Gamma_1}) \setminus I_{\Gamma_2}$$

with[8] $\alpha_i(1) \in \Gamma_1$, $\alpha_i(2) \in \Gamma_2$ and $\alpha_i((1,2)) \subset I_{\Gamma_1} \setminus (\Gamma_2 \cup I_{\Gamma_2})$.

---

[7] On the word "homotopy" see footnote on p. 43.

[8] For example, fix $z_0 \in I_{\Gamma_2}$ and take a ray from z_0 which is not tangent to the Γ_i; then use the segment between the closest point on Γ_1 and the farthest point on Γ_2 as α_1; now rotate the line by an angle small enough so that it does not become tangent to the Γ_i's and take again the segment between the closest point on Γ_1 and the farthest point on Γ_2 as α_2.

Now we denote:

- γ_j the arc on $\mathit{\Gamma}_j$ from $\alpha_1(j)$ to $\alpha_2(j)$ and
- $\tilde{\gamma}_j$ the arc on $\mathit{\Gamma}_j$ from $\alpha_2(j)$ to $\alpha_1(j)$.

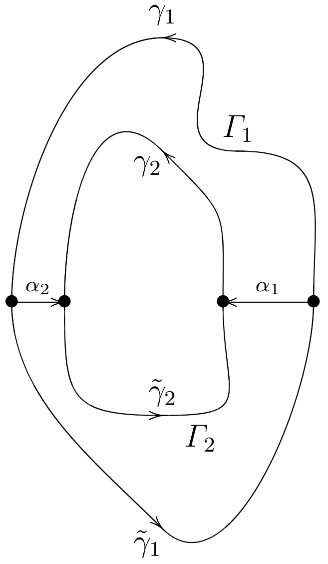
Then, by Cauchy's Theorem,

$$0 = \int_{\gamma_1+\alpha_2-\gamma_2-\alpha_1} f \,\mathrm{d}z,$$
$$0 = \int_{\tilde{\gamma}_1+\alpha_1-\tilde{\gamma}_2-\alpha_2} f \,\mathrm{d}z.$$

Adding up, we obtain

$$\begin{aligned} 0 &= \int_{\gamma_1} f \,\mathrm{d}z + \int_{\tilde{\gamma}_1} f \,\mathrm{d}z + \int_{-\gamma_2} f \,\mathrm{d}z + \int_{-\tilde{\gamma}_2} f \,\mathrm{d}z \\ &= \int_{\mathit{\Gamma}_1} f \,\mathrm{d}z - \int_{\mathit{\Gamma}_2} f \,\mathrm{d}z \end{aligned}$$

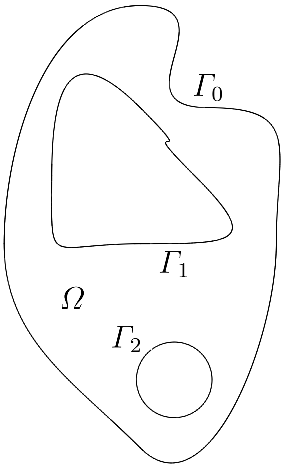
as desired. □



Remark. The homotopy version of Cauchy's Theorem easily generalizes to multiple simple closed contours: Let $f : D \to \mathbb{C}$ be holomorphic and $\mathit{\Omega} \subset D$ bounded with piecewise regular boundary components $\mathit{\Gamma}_0, \dots \mathit{\Gamma}_n \subset D$ such that $\mathit{\Omega} \subset I_{\mathit{\Gamma}_0}$ is in the interior of $\mathit{\Gamma}_0$. Then

$$\int_{\mathit{\Gamma}_0} f \,\mathrm{d}z = \sum_{k=1}^{n} \int_{\mathit{\Gamma}_k} f \,\mathrm{d}z.$$

ex Formulate and prove the homotopy version of Cauchy's theorem for multiple simple closed contours. *Hint:* Use a drawing for the proof – is this a proof then?



Example. We use Cauchy's Theorem, partial fractions and the homotopy version of Cauchy's Theorem for multiple simple closed contours to compute:

$$\begin{aligned} \int_{|z|=2} \frac{\mathrm{d}z}{z^2-1} &= \int_{|z-1|=\frac{1}{2}} \frac{\mathrm{d}z}{z^2-1} + \int_{|z+1|=\frac{1}{2}} \frac{\mathrm{d}z}{z^2-1} \\ &= \frac{1}{2}\int_{|z-1|=\frac{1}{2}} \frac{\mathrm{d}z}{z-1} - \frac{1}{2}\int_{|z-1|=\frac{1}{2}} \frac{\mathrm{d}z}{z+1} + \frac{1}{2}\int_{|z+1|=\frac{1}{2}} \frac{\mathrm{d}z}{z-1} - \frac{1}{2}\int_{|z+1|=\frac{1}{2}} \frac{\mathrm{d}z}{z+1} \\ &= \frac{1}{2}\int_{|z-1|=\frac{1}{2}} \frac{\mathrm{d}z}{z-1} - 0 + 0 - \frac{1}{2}\int_{|z+1|=\frac{1}{2}} \frac{\mathrm{d}z}{z+1} \\ &= i\pi - i\pi = 0. \end{aligned}$$

ex Compute $4 \cdot \int_{\mathit{\Gamma}} \frac{(1-i)z-(1+i)}{z^3-3z^2-z+3} \,\mathrm{d}z$, where

$$\mathit{\Gamma} = \left\{ z = x + iy \in \mathbb{C} \mid \left(\frac{x}{2}\right)^2 + \left(\frac{y}{7}\right)^2 = 1 \right\}$$

(an ellipse with half-axes radii 2 and 7).

(ex) We again explore *winding numbers* (compare exercise on p. 41).

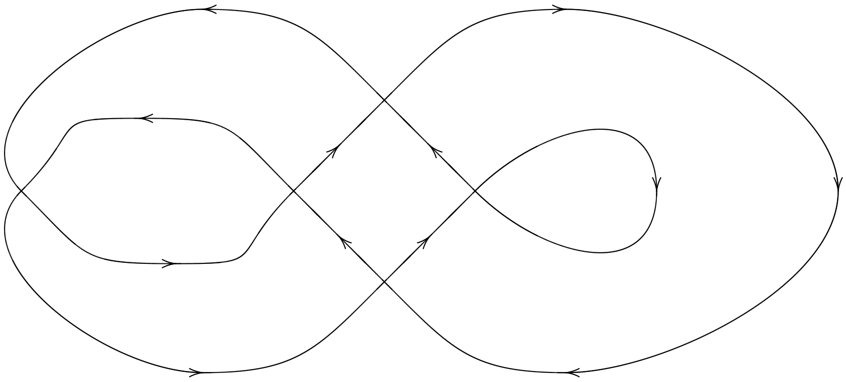
(i) Let $\gamma : [a, b] \to \mathbb{C}$ be a simple closed regular path, so that $\Gamma = \gamma([a, b])$ is oriented counter-clockwise as usual. Prove that, for $z \notin \Gamma$,

$$w(\gamma, z) = \begin{cases} 1 & \text{if } z \in I_\Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Deduce that, if a domain $D \subset \mathbb{C}$ is simply connected, then $w(\gamma, z) = 0$ for all $z \notin D$ and simple closed regular paths $\gamma : [a, b] \to D$ (compare [ST, Section 8.7]).

Note: This turns out to be also a sufficient condition but this is rather hard to prove in our setup.

(ii) "Compute" the winding numbers for all points $z \in \mathbb{C} \setminus \Gamma$ for the following (oriented) contour "by inspection" (and/or "educated guessing"):



III.3. Cauchy Formulae

References: [DET, Section 3.6] and [ST, Sections 10.1 – 10.4]
(Cauchy's Formula: [DET, Theorem 3.6.1] and [ST, Lemma 10.1]; Cauchy's Formulae for Derivatives: [DET, Theorem 3.6.2] and [ST, Theorem 10.3]; Morera's Theorem: [DET, Theorem 3.6.7] and [ST, Theorem 10.4]; Cauchy's Inequalities: [DET, Theorem 3.6.3] and [ST, Lemma 10.5])

For what follows we assume that we have the general form of Cauchy's Theorem available (so that we can drop any assumptions "with continuous f'" or "where D is a star-domain").

As a first impressive application of (the homotopy version of) Cauchy's Theorem, we learn that a holomorphic function is known in a region as soon as it is known on its boundary:

Theorem III.3.1 (Cauchy's Formula). *Let $f : D \to \mathbb{C}$ be holomorphic and $\Gamma \subset D$ a simple closed contour so that its interior $I_\Gamma \subset D$. Then, for all $z_0 \in I_\Gamma$,*

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} \,\mathrm{d}z.$$

Proof. Fix $z_0 \in I_\Gamma$ and $\varepsilon > 0$.

Since I_Γ is open and f continuous at z_0, we can fix $\delta > 0$ s.t.

$$B_{2\delta}(z_0) \subset I_\Gamma \qquad \text{and} \qquad |z - z_0| < 2\delta \Rightarrow |f(z) - f(z_0)| < \varepsilon.$$

Now, by the Homotopy version of Cauchy's Theorem

$$\int_\Gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z = \int_{|z-z_0|=\delta} \frac{f(z)}{z - z_0}\,\mathrm{d}z$$

and, since $\int_{|z-z_0|=\delta} \frac{\mathrm{d}z}{z-z_0} = 2\pi i$ (used in the step "$\star$") and by the ML-inequality,

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z - f(z_0)\right| &\overset{\star}{=} \left|\frac{1}{2\pi i}\int_{|z-z_0|=\delta} \frac{f(z) - f(z_0)}{z - z_0}\,\mathrm{d}z\right| \\ &\overset{ML\text{-ineq.}}{\leq} \frac{1}{2\pi}\max_{|z-z_0|=\delta}\left|\frac{f(z) - f(z_0)}{z - z_0}\right| 2\pi\delta \\ &\overset{|z-z_0|=\delta}{=} \max_{|z-z_0|=\delta}|f(z) - f(z_0)| \\ &\overset{\partial B_\delta(z_0)\subset B_{2\delta}(z_0)}{<} \varepsilon.\end{aligned}$$

Thus $|\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z - f(z_0)| = 0$. □

If we now knew that we could differentiate under the integral we would obtain the formulae

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(w)}{(w - z)^{n+1}}\,\mathrm{d}w$$

for the derivatives of f. However, can we really interchange differentiation and integration? The answer is "yes" by Lemma III.2.6 (that we have already used in Theorem III.2.5).

Remark. (Not examinable!)
Even without Lemma III.2.6 we are able to prove Theorem III.3.2 by taking the following different route:

> **Lemma.** *Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise regular path and let $\Gamma = \gamma([a, b])$; suppose $\varphi : \Gamma \to \mathbb{C}$ is continuous. For $n \in \mathbb{N}$ define*
>
> $$f_n : \mathbb{C} \setminus \Gamma \to \mathbb{C}, \quad f_n(z) = \int_\gamma \frac{\varphi(w)}{(w - z)^n}\,\mathrm{d}w.$$
>
> *Then each f_n is holomorphic with $f_n' = n\, f_{n+1}$.*
>
> For the proof of this lemma we shall use the following:
> **Lemma.** *Let $\psi : B_r(z_0) \to \mathbb{C}$ be twice (complex) differentiable with continuous ψ''. Then, for $z \in B_r(z_0)$,*
>
> $$\psi(z) - \psi(z_0) - \psi'(z_0)(z - z_0) = \int_{[z_0,z]} \psi''(\tau)(z - \tau)d\tau.$$
>
> *In particular,*
>
> $$|\psi(z) - \psi(z_0) - \psi'(z_0)(z - z_0)| \leq \max_{\tau\in[z_0,z]} |\psi''(\tau)|\,|z - z_0|^2.$$

Proof. Consider

$$\tau \mapsto g(\tau) = \psi(z) - \psi(\tau) - \psi'(\tau)(z-\tau).$$

This function is differentiable with

$$g'(\tau) = -\psi''(\tau)(z-\tau);$$

hence, by the FTC for path integrals,

$$g(z_0) = g(z) - \int_{[z_0,z]} g'(\tau)\,\mathrm{d}\tau = \int_{[z_0,z]} \psi''(\tau)(z-\tau)d\tau.$$

From this

$$\begin{aligned}|g(z_0)| &\leq \max_{\tau\in[z_0,z]} |\psi''(\tau)(z-\tau)|\,|z-z_0| \\ &\leq \max_{\tau\in[z_0,z]} |\psi''(\tau)|\,|z-z_0|^2\end{aligned}$$

by the ML-inequality. ☐

Proof of the above Lemma. Consider $z_0 \in D = \mathbb{C} \setminus \Gamma$; since D is open there is $\varrho > 0$ so that $B_{2\varrho}(z_0) \subset D$. Then $|w-z| > \varrho$ for all $z \in B_\varrho(z_0)$.

Let $\psi(z) = \frac{1}{(w-z)^n}$. Then $\psi'(z) = \frac{n}{(w-z)^{n+1}}$ and $\psi''(z) = \frac{n(n+1)}{(w-z)^{n+2}}$ and, by the previous auxiliary lemma,

$$\begin{aligned}\left|\frac{\psi(z)-\psi(z_0)-\psi'(z_0)(z-z_0)}{z-z_0}\right| &\leq \max_{\tau\in[z_0,z]}\left|\frac{n(n+1)}{(w-\tau)^{n+2}}\right| |z-z_0| \\ &\leq \frac{n(n+1)}{\varrho^{n+2}}\,|z-z_0|\end{aligned}$$

for any $z \in B_\varrho(z_0)$ and all $w \in \Gamma$.

Now take $z \in B_\varrho(z_0)$; then, using the ML-inequality again,

$$\begin{aligned}&\left|\frac{f_n(z)-f_n(z_0)}{z-z_0} - nf_{n+1}(z_0)\right| \\ &\qquad = \left|\int_\Gamma \varphi(w)\left\{\frac{1}{(z-z_0)}\left(\frac{1}{(w-z)^n} - \frac{1}{(w-z_0)^n}\right) - \frac{n}{(w-z_0)^{n+1}}\right\}\mathrm{d}w\right| \\ &\qquad = \left|\int_\Gamma \varphi(w)\ \frac{\psi(z)-\psi(z_0)-\psi'(z_0)(z-z_0)}{z-z_0}\ \mathrm{d}w\right| \\ &\qquad \leq \max_{w\in\Gamma}\left|\varphi(w)\ \frac{\psi(z)-\psi(z_0)-\psi'(z_0)(z-z_0)}{z-z_0}\right| L \\ &\qquad \leq \max_{w\in\Gamma}|\varphi(w)|\frac{n(n+1)L}{\varrho^{n+2}}\,|z-z_0| \quad \to \quad 0 \quad \text{as} \quad z \to z_0.\end{aligned}$$

Thus f_n is (complex) differentiable at z_0 with $f_n'(z_0) = nf_{n+1}(z_0)$. ☐

Theorem III.3.2 (Cauchy's Formulae for the Derivatives)**.** *Let $f : D \to \mathbb{C}$ be holomorphic and let $\Gamma \subset D$ be a simple closed contour so that its interior $I_\Gamma \subset D$. Then, for all $z \in I_\Gamma$ and $n \in \mathbb{N}$,*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^{n+1}}\,\mathrm{d}w.$$

Proof. This follows now immediately from Lemma III.2.6 (or the lemma in the previous remark). ☐

Corollary III.3.3. *Let $f : D \to \mathbb{C}$ be holomorphic. Then $f : D \to \mathbb{C}$ has derivatives $f^{(n)}$ of all orders $n \in \mathbb{N}$ in D.* ☐

Remark. This shows that Corollary II.3.4 (real and imaginary parts of a holomorphic function are harmonic) holds without the additional assumption of f being twice continuously differentiable.

Corollary III.3.4 (Morera's Theorem)**.** *Let $f : D \to \mathbb{C}$ be continuous so that $\int_\Gamma f(z)\,\mathrm{d}z = 0$ for any closed contour $\Gamma \subset D$. Then f is holomorphic.*

Proof. Let $f : D \to \mathbb{C}$ be continuous and $\int_\Gamma f(z)\,\mathrm{d}z = 0$ for every closed contour $\Gamma \subset D$.

By the criterion for the FTC (Lemma III.2.1), f has an anti-derivative $F : D \to \mathbb{C}$. In particular, F is holomorphic with $F' = f$.

But, since F is holomorphic, it has derivatives $F^{(n)}$ of any order in D, in particular, $f' = F''$ exists. ☐

Corollary III.3.5 (Cauchy's inequalities)**.** *Let $f : D \to \mathbb{C}$ be holomorphic and $\Gamma = \{z \in \mathbb{C} \mid |z - z_0| = R\} = \partial B_R(z_0) \subset D$ so that $I_\Gamma = \{z \in \mathbb{C} \mid |z - z_0| < R\} = B_R(z_0) \subset D$. Then, for any $n \in \mathbb{N}$,*

$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n} \max_{z\in\Gamma} |f(z)|.$$

Proof. By Cauchy's formulas for the derivatives

$$\begin{aligned} |f^{(n)}(z_0)| &= \left| \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z \right| \\ &\leq \frac{n!}{2\pi} \max_{z\in\Gamma} \frac{|f(z)|}{|z - z_0|^{n+1}}\, 2\pi R \\ &= \frac{n!}{R^n} \max_{z\in\Gamma} |f(z)|, \end{aligned}$$

where the ML-inequality provides the estimate. ☐

III.4. Applications

References: [DET, Section 3.6] and [ST, Sections 10.4 & 10.7]
(Liouville's Theorem: [DET, Theorem 3.6.4] and [ST, Theorem 10.6]; Gauss' Fundamental Theorem of Algebra: [DET, Theorem 3.6.5] and [ST, Theorem 10.7]; Local Maximum Modulus Theorem: [ST, Proposition 10.12]; Potential theory (physics): [DET, Chapter 6] and [ST, Section 13.4]; Logarithms and complex powers: [DET, Section 2.4] and [ST, Sections 7.3 & 14.5 & 14.6])

III.4.1. Theoretical Applications

Definition. An *entire* function is a function that is holomorphic in all of $\mathbb{C}$.

Theorem III.4.1 (Liouville's Theorem)**.** *A bounded entire*[9] *function is constant.*

Proof. Suppose $\forall z \in \mathbb{C} : |f(z)| \leq M$. Fix $z \in \mathbb{C}$ arbitrarily. Then, by Cauchy's inequalities,

$$|f'(z)| \leq \frac{1}{R} M$$

for any $R > 0$ (here we use that f is entire). Since R is arbitrary, we find $f'(z) = 0$ and since z was arbitrary, $f' \equiv 0$. Hence $f \equiv \text{const}$. ☐

Theorem III.4.2 (Gauss' Fundamental Theorem of Algebra)**.** *Every non-constant polynomial has a zero in* $\mathbb{C}$.

(ex) Prove Gauss' Fundamental Theorem of Algebra.
Hint: suppose that a polynomial $p(z)$ has no zeros and conclude that $f(z) = \frac{1}{p(z)}$ is bounded.

Corollary III.4.3. *Every non-constant polynomial can be factorized,*

$$p(z) = c(z - z_1) \cdots (z - z_n)$$

with the zeros $z_1, \ldots, z_n \in \mathbb{C}$ *of* $p(z)$.

Proof. This can be proved by induction on the degree (ex). ☐

Theorem III.4.4 (The local Maximum Modulus Theorem)**.** *Let* $f : B_R(z_0) \to \mathbb{C}$ *be holomorphic and assume that* $|f(z)| \leq |f(z_0)|$ *for all* $z \in B_R(z_0)$. *Then* f *is constant.*

Proof. Fix $r \in (0, R)$. By Cauchy's formula

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{z - z_0} \, \mathrm{d}z \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + r\,\mathrm{e}^{it})}{r\,\mathrm{e}^{it}} \, ir\,\mathrm{e}^{it} \, \mathrm{d}t \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\,\mathrm{e}^{it}) \, \mathrm{d}t. \end{aligned}$$

By the triangle inequality (for real integrals) and since $|f(z)| \leq |f(z_0)|$ for all z with $|z - z_0| = r$ we find

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + r\,\mathrm{e}^{it})| \, \mathrm{d}t \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| \, \mathrm{d}t = |f(z_0)|.$$

---

[9] Recalling that the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is compact (and that the continuous image of a compact set is compact, and thus bounded in $\mathbb{C}$), Liouville's Theorem has the following consequence: The only holomorphic functions $f : \hat{\mathbb{C}} \to \mathbb{C}$ are the constants.

Thus, equality holds throughout the previous line; this gives

$$\int_0^{2\pi} |f(z_0)| - |f(z_0 + r\,e^{it})|\;dt = 0 \qquad \Rightarrow \qquad \forall t \in [0, 2\pi] : |f(z_0)| - |f(z_0 + r\,e^{it})| = 0$$

since $|f(z_0)| - |f(z_0 + r\,e^{it})| \geq 0$ and continuous (noting that if $\int_a^b g(x)\,dx = 0$ for a nonnegative real continuous function g then $g \equiv 0$ (ex)).

Since this holds for all $r \in (0, R)$ we conclude that $|f|$, hence f (see p. 23), is constant on $B_R(z_0)$. ☐

III.4.2. Practical Applications

Cauchy's Formulas enlarge the possibilities for "calculation-free integration":

(i) $\int_{|z|=1} \frac{e^z\,dz}{z^3} = \frac{2\pi i}{2!} \frac{d^2}{dz^2}\Big|_{z=0} e^z = \pi i$;

(ii) $\int_{|z-i|=1} \frac{z^2}{z^2+1} = \int_{|z-i|=1} \frac{z^2/(z+i)}{(z-i)^1} = 2\pi i\, \frac{z^2}{z+i}\Big|_{z=i} = -\pi$;

(iii) $\int_{|z|=1} e^z \operatorname{Re} z\;dz = \int_{|z|=1} \frac{z\,e^z}{2} + \frac{e^z}{2z}\,dz = 0 + 2\pi i \frac{e^z}{2}\Big|_{z=0} = \pi i$, since $z\bar{z} = 1 \Leftrightarrow \bar{z} = \frac{1}{z}$ so that $\operatorname{Re} z = \frac{1}{2}(z + \frac{1}{z})$ whenever $|z| = 1$.

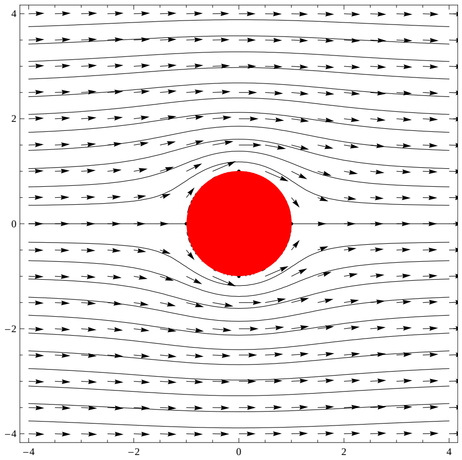
(ex) Evaluate $\int_{|z|=1} \frac{e^{i\alpha z}}{(z-z_0)^2}\,dz$ for $\alpha > 0$ and

$$\text{(i)} \quad |z_0| < 1, \qquad \text{(ii)} \quad |z_0| > 1.$$

III.4.3. Physical Applications (Fluid Dynamics) (Not examinable!)

We look at harmonic functions again (compare Corollary II.3.4), and first show why they are studied in physics (fluid dynamics/electrostatics):

A complex function $\tilde{f}(z)$ defines a two-dimensional vector field $(u, v, 0)$ which represents a steady-state (there is no time-dependence), laminar (stratified into layers[10]), incompressible (in mathematical terms: $\operatorname{div} \tilde{f} = \nabla \cdot \tilde{f} = \tilde{u}_x + \tilde{v}_y = 0$) and irrotational (in mathematical terms: $\operatorname{curl} \tilde{f} = \nabla \times \tilde{f} = (\tilde{v}_x - \tilde{u}_y) e_z = 0$; there are no vortices ("whirlpools")) fluid flow over a domain D precisely when the conjugate of $\tilde{f}(z)$ is holomorphic. In electrostatics, $\tilde{f}(z)$ describes the electrical vector field in a charge-free domain D (or the charge-free part of the domain D).



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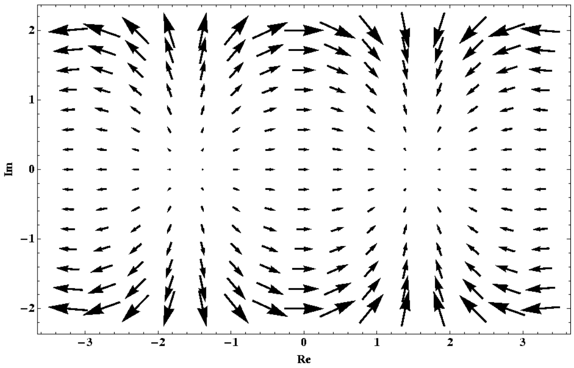
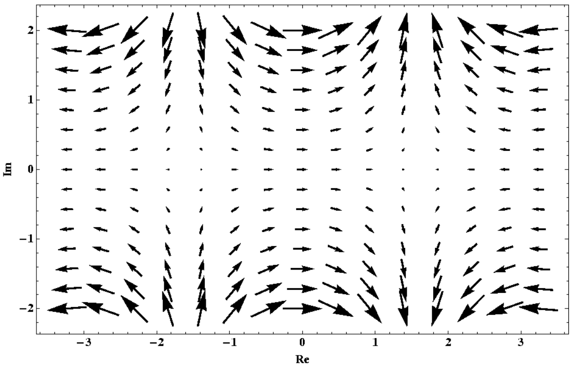
[10] If you want to see what the effect of "laminar" is, look at the following physics video:

http://www.youtube.com/watch?v=p08_KlTKP50

On p. 25, we have already compared the vector fields for for $f(z) = \cos(z)$ (see below on the left) and $f(z) = \overline{\cos(z)}$ (see below on the right). Here, we note that the vector field on the right satisfies the conditions just stated (e.g., it has no "sinks" and "springs" where the incompressibility assumptions would certainly not hold).





Remark. For a holomorphic function f, if we would assume continuity of f', then Poincaré's Lemma would yield an alternative proof for the existence of an anti-derivative of f.

Lemma (Poincaré's Lemma)**.** *Let $\vec{v} = (\alpha, \beta) : D \to \mathbb{R}^2$ be a continuously differentiable vector field on a star-domain D such that $\alpha_y = \beta_x$ (i.e., $\operatorname{curl} \vec{v} = 0$). Then $\vec{v}$ has a* potential *$\varphi : D \to \mathbb{R}$, i.e., φ is (real) differentiable with $\nabla \varphi = \vec{v}$.*

Lemma. *If $f : D \to \mathbb{C}$ is holomorphic with continuous f' in a star-domain D, then it has an anti-derivative $F : D \to \mathbb{C}$.*

Proof. Write $f = u + iv$; we are seeking $F = U + iV$ so that

$$F' = U_x + iV_x = V_y - iU_y = u + iv \qquad \Leftrightarrow \qquad \begin{cases} \nabla U = (u, -v) & \text{and} \\ \nabla V = (v, u) \end{cases}$$

Now, since f is holomorphic,

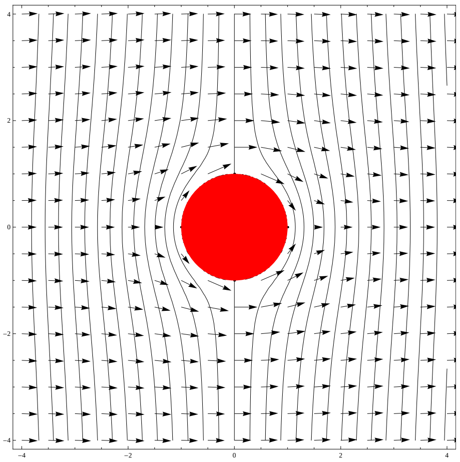
$$\begin{aligned} u_y = -v_x \qquad &\Rightarrow \qquad \exists U : D \to \mathbb{R} : \nabla U = (u, -v) \\ v_y = u_x \qquad &\Rightarrow \qquad \exists V : D \to \mathbb{R} : \nabla V = (v, u) \end{aligned}$$

by Poincaré's Lemma since the partial derivatives of u and v are continuous and D is a star-domain.

Finally, U and V are by their gradients determined up to a real constant each. These constants combine to determine F up to a complex constant. ☐

In physics, a vector field that possesses a potential is called *conservative* or *exact*. In that case, the work exerted to move a particle along a path, only depends on the endpoints of the path (the difference of the potential at the endpoints).

Also note that the functions U and V in the previous proof are harmonic conjugate to each other. In the picture at the beginning of this section, we showed a vector field of a fluid flowing around a unit disk. In that picture, the *stream-lines* are shown (which are the level curves $V(x, y) = \text{const.}$), i.e., the paths along which the fluid (or "fluid particle") flows. On the figure on the right, we depicted the *equipotential lines* (the level curves $U(x, y) = \text{const.}$) which shows us how the fluid is advancing in each unit of time.



We can also use Poincaré's Lemma to prove a converse of Corollary II.3.4.

Lemma. *Let $u : D \to \mathbb{R}$ be a twice continuously (partial) differentiable harmonic function on a star-domain D. Then u is the real part of a holomorphic function.*

Proof. We let $\alpha = -u_y$ and $\beta = u_x$. Then α and β are continuously differentiable with $\beta_y - \alpha_x = \Delta u = 0$ so that, by Poincaré's Lemma, there is $v : D \to \mathbb{R}$ with $v_x = \alpha = -u_y$ and $v_y = \beta = u_x$.

Now, as u and v are both continuously differentiable and satisfy the Cauchy-Riemann equations $f = u + i\,v$ is holomorphic by the sufficient Cauchy-Riemann conditions. $\square$

ⓔⓧ Let $u(x, y) = xy$ on $D = \mathbb{C}$. Show that u is harmonic and find a holomorphic function f with $\operatorname{Re} f = u$.

Solution: Clearly $\Delta u = u_{xx} + u_{yy} = 0 + 0 = 0$ and observe that $u(x, y) = -\operatorname{Re}(\frac{iz^2}{2} - 42i)$.

III.4.4. Logarithms & Multifunctions

Together with the exponential function, we must also study its inverse function, the (or, more precisely, a) logarithm.

Definition. Let D be a simply connected domain with $0 \notin D$. A holomorphic function $F : D \to \mathbb{C}$ is a *logarithm* if $\mathrm{e}^{F(z)} = z$ for all $z \in D$.
On the cut plane $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, the function $\operatorname{Log} z = \log|z| + i \arg(z)$ with $\arg(z) \in (-\pi, \pi)$ (and where log denotes the real logarithm) is a logarithm, called the *principal value of the logarithm.*

Lemma III.4.5. *Let D be a simply connected domain with $0 \notin D$ and F be a logarithm on D. Then $\tilde{F}$ is a logarithm on D iff $\tilde{F} = F + 2\pi i\,k$ for some $k \in \mathbb{Z}$.*

Proof. "$\Leftarrow$": Clearly, $\mathrm{e}^{2\pi i\,k} = 1$ for $k \in \mathbb{Z}$.
"$\Rightarrow$": From $\mathrm{e}^{F(z)} = z = \mathrm{e}^{\tilde{F}(z)}$, and thus $\mathrm{e}^{F(z) - \tilde{F}(z)} = 1$, we conclude that for every $z \in \mathbb{C} \setminus \{0\}$ there is a $k(z) \in \mathbb{Z}$ such that $F(z) - \tilde{F}(z) = 2\pi i\,k(z)$. But $k(z)$ is a continuous, integer-valued function on the connected set D and therefore a constant. $\square$

Lemma III.4.6. *Let D be a simply connected domain with $0 \notin D$. F is a logarithm on D iff $F'(z) = \frac{1}{z}$ on D and $\mathrm{e}^{F(a)} = a$ for at least one $a \in D$.*

Proof. "$\Rightarrow$": From $e^{F(z)} = z$ and the chain rule follows $1 = F'(z)\, \mathrm{e}^{F(z)} = F'(z) \cdot z$, and so $F'(z) = \frac{1}{z}$.
"$\Leftarrow$": Let F be an anti-derivative of $z \mapsto \frac{1}{z}$ with $\mathrm{e}^{F(a)} = a$ for one $a \in D$. Then, the function $g(z) = z\, \mathrm{e}^{-F(z)}$ is holomorphic on D and satisfies $g' \equiv 0$ on D. Thus g is constant on D, in fact $g \equiv 1$, and it follows that $e^{F(z)} = z$ for all $z \in D$. □

Remark. The principal value of the logarithm $\operatorname{Log} z$ coincides on $\mathbb{R}_{>0} \subset \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ with the usual real logarithm.

Recall that $z \mapsto \frac{1}{z}$ has no anti-derivative in any punctured neighbourhood of the origin. Therefore, we cannot impose a restriction which determines the logarithm uniquely and simultaneously allow z to move freely in $\mathbb{C} \setminus \{0\}$ with the logarithm varying continuously. One possibility to obtain *one* inverse function of the exponential function is to extend the definition of "function". We say that $\log z = \log |z| + i \arg(z) + 2\pi i\, k$ (with $k \in \mathbb{Z}$ and $\arg(z) \in (-\pi, \pi]$) is a *multi-valued function* or *multifunction* with infinitely many *branches*, each for a different k. Each branch is a (single-valued) holomorphic function in every simply connected domain D.

Example. The possible logarithms of $\frac{1}{\sqrt{2}}(1+i) = \mathrm{e}^{i\pi/4}$ are $i\pi\left(\frac{1}{4} + 2\,k\right)$, $k \in \mathbb{Z}$. The principal value of the logarithm is $\operatorname{Log}\left(\frac{1}{\sqrt{2}}(1+i)\right) = i\pi/4$.

Remark. Defining the argument of z by $\arg z = \{\varphi \in \mathbb{R} | z = |z|\, \mathrm{e}^{i\varphi}\}$, we see that arg is also a multifunction, its principal value Arg is given by the above restriction $\arg(z) \in (-\pi, \pi]$.

ex Noting that for the multifunction log on the cut plane $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$we have $\log z = \log |z| + i \operatorname{Arg} z + 2k\,\pi i$ and $\log(-z) = \log|z| + i \operatorname{Arg} z + (2k+1)\,\pi i$ where $k \in \mathbb{Z}$, find the mistake in the following argument:

$$\begin{aligned} & \log\left((-z)^2\right) = \log\left(z^2\right) \\ \Rightarrow \quad & \log(-z) + \log(-z) = \log z + \log z \\ \Rightarrow \quad & 2\log(-z) = 2\log z \\ \Rightarrow \quad & \log(-z) = \log z. \end{aligned}$$

We can now also define arbitrary (complex) powers of arbitrary (complex) numbers.

Definition. If a is a complex number, we define for $z \neq 0$,

$$z^a = \mathrm{e}^{a \log z}\,.$$

The *principal value* of z^a is $\mathrm{e}^{a \operatorname{Log} z}$.

Example.

$$1^{\sqrt{2}} = \mathrm{e}^{\sqrt{2} \log 1} = \mathrm{e}^{\sqrt{2}\, 2\pi i\, k} = \cos(2\pi k\, \sqrt{2}) + i\, \sin(2\pi k\, \sqrt{2}) \qquad (\text{with } k \in \mathbb{Z}).$$

Note that the values of $1^{\sqrt{2}}$ are dense in the unit circle $\partial B_1(0)$.

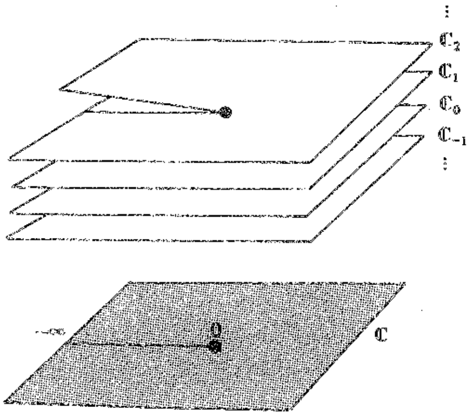
Example. There are two branches of the square root in $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$ (i.e., two functions f such that $(f(z))^2 = z$):

$$z^{\frac{1}{2}} = \mathrm{e}^{\frac{1}{2}\log z} = \mathrm{e}^{\frac{1}{2}(\mathrm{Log}\, z + 2\pi i k)} = \mathrm{e}^{\frac{1}{2}\,\mathrm{Log}\, z}\, \mathrm{e}^{\pi i\, k)} = \pm\, \mathrm{e}^{\frac{1}{2}\,\mathrm{Log}\, z}\,.$$

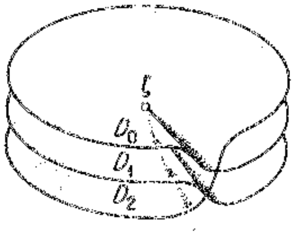
ⓔⓧ Calculate all possible values of i^i. What is the principal value of i^i?
Do the same for $i^{7/10}$.
State (without proof) under which conditions the power b^a is single-valued, has finitely many values or has infinitely many values.

ⓔⓧ Determine the inverse function arccos of the complex cosine $\cos z = \frac{1}{2}\,(\mathrm{e}^{i\,z} + \mathrm{e}^{-i\,z})$. Hence calculate all values $\arccos(3/2)$. Which one of these is the principle value?

Remark. (Not examinable!) Another way out of the above dilemma is to define the logarithm (and then also other multifunctions) on a more complicated surface, the so-called *Riemann surface*, consisting of infinitely many planes joined together so that the function varies continuously as one passes from one plane to the next, see [DET, Section 2.6] and [ST, Sections 14.5 & 14.6].

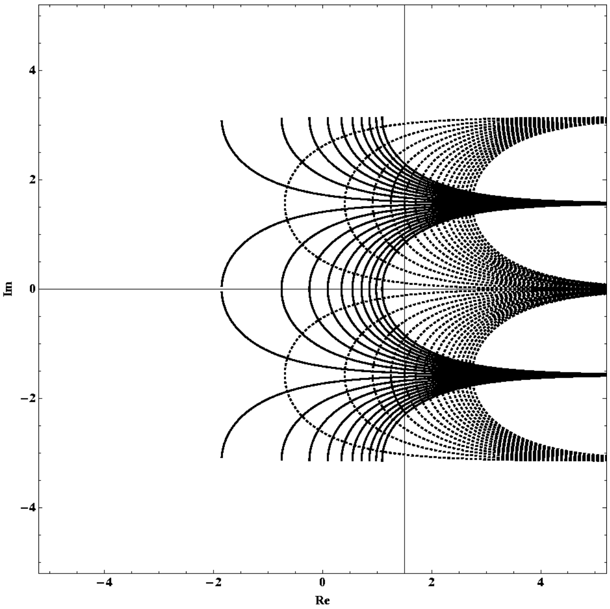


Riemann surface for the logarithm: countably infinitely many copies $\mathbb{C}_k$ (where $k \in \mathbb{Z}$, see above) of the cut plane (below) are pasted together "at the respective cut" to allow the logarithm to vary continuously. This figure is taken from [ST, Fig. 14.8].

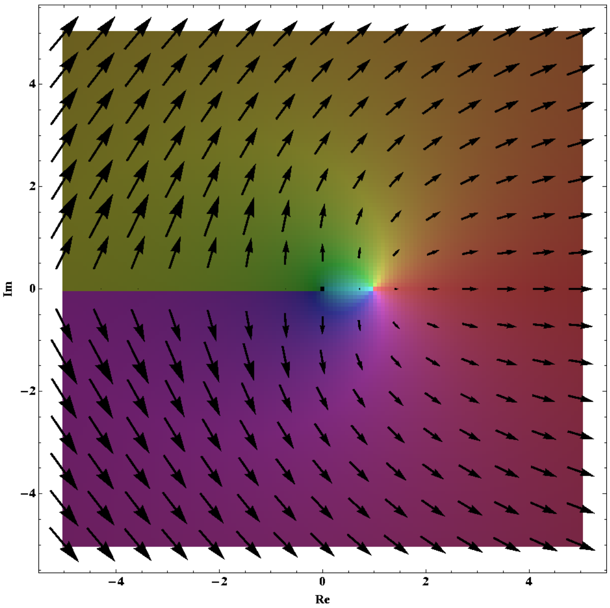


Riemann surface for the cubic root $z \mapsto \sqrt[3]{z}$: The cubic root has three branches that are pasted together in such a way that the principal branch D_0 connects to the second branch D_1, the second to the third branch D_2 and the third to the principal branch again. This picture is taken from R.A. Silverman: Introductory Complex Analysis; Dover, NY (1972); library: `513.317 SIL` (Fig. 16.6.).

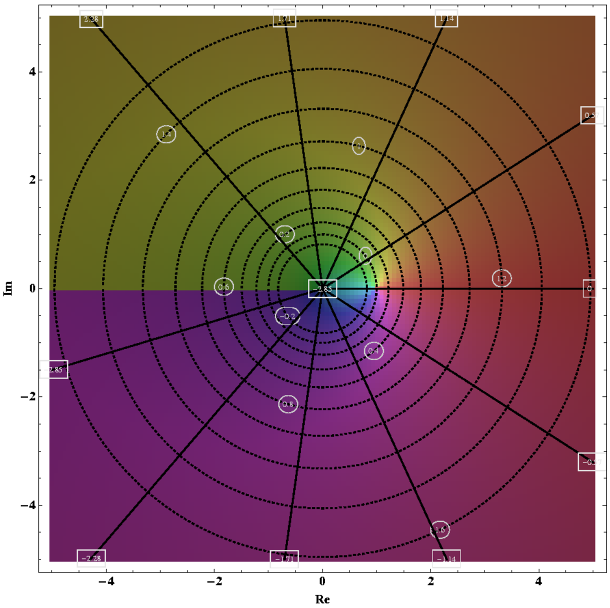
The principal value of the logarithm $\mathrm{Log}(z)$**:**



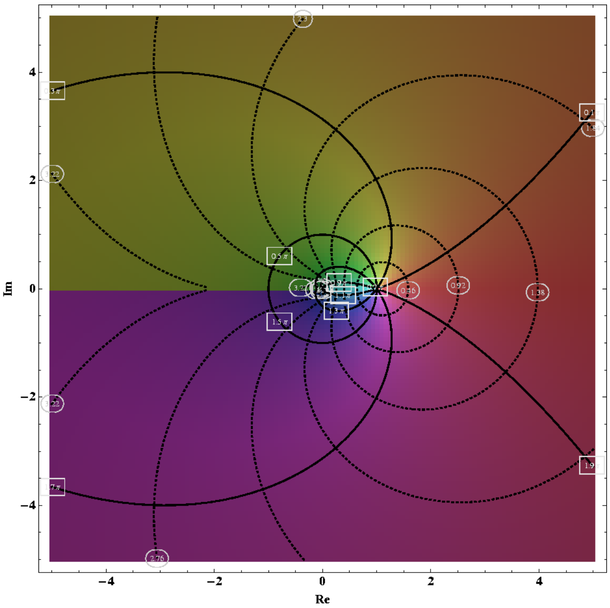
The image of the vertical (solid) and horizontal (dotted) lines under $f(z) = \mathrm{Log}(z)$; they are contained in the strip $\{z \in \mathbb{C} \mid \mathrm{Im}\, z \in (-\pi, \pi]\}$.



Vector field $(\mathrm{Re}\,\mathrm{Log}(z), \mathrm{Im}\,\mathrm{Log}(z))$ and colouring of the complex plane for $\mathrm{Log}(z)$.

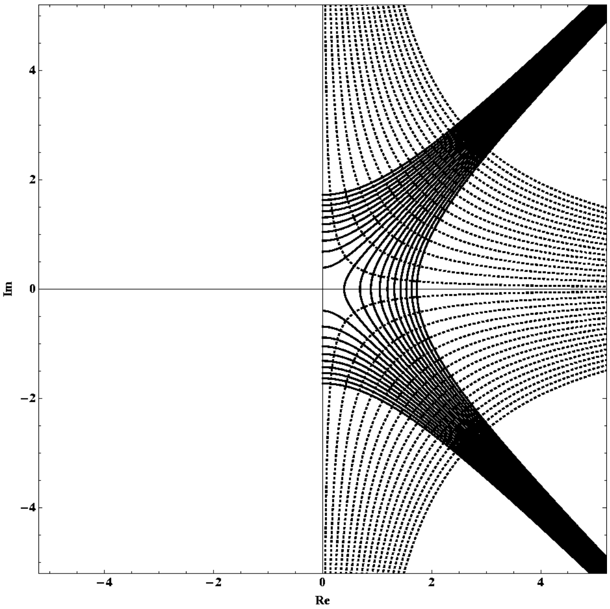


Colouring and level curves of constant real (dotted) and imaginary (solid) part for $\mathrm{Log}(z)$; they intersect perpendicularly.

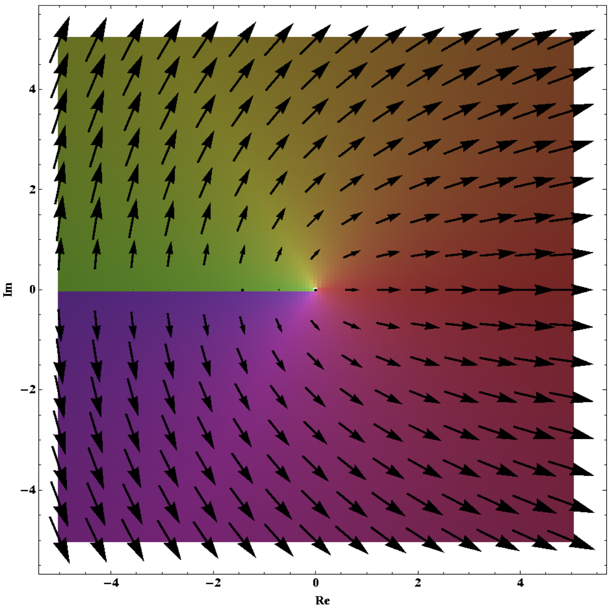


Colouring and level curves of constant modulus (dotted) and argument (solid) for $\mathrm{Log}(z)$; they intersect perpendicularly.

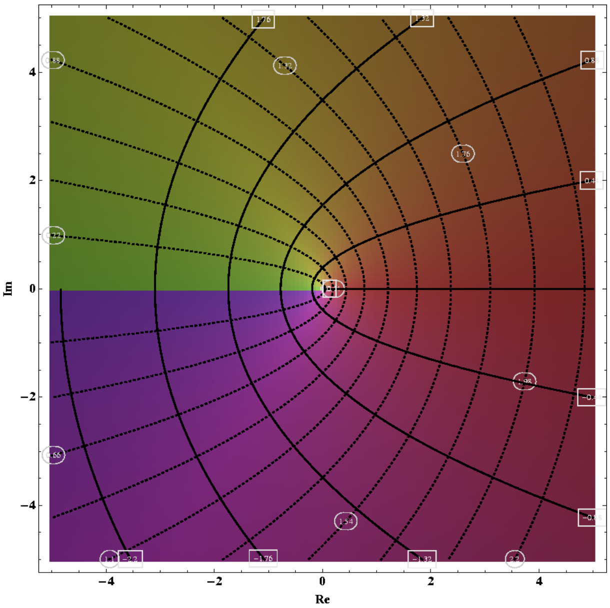
The principal value of the square root $\sqrt{z} = z^{\frac{1}{2}} = \mathrm{e}^{\frac{1}{2}\operatorname{Log}(z)}$:



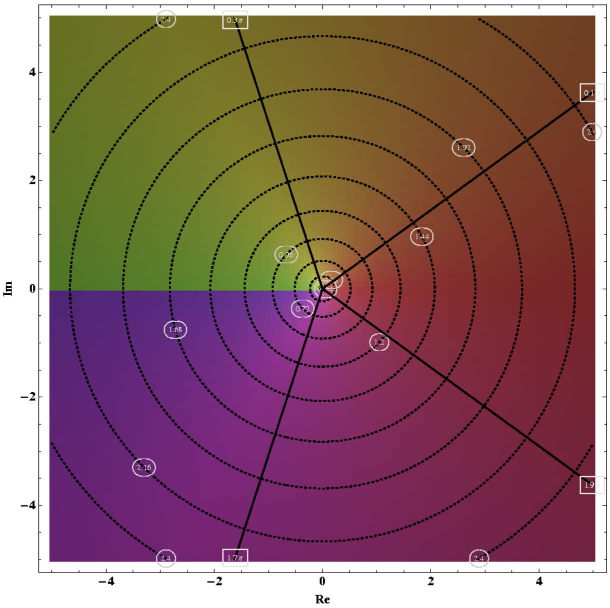
The image of the vertical (solid) and horizontal (dotted) lines under $f(z) = \sqrt{z}$; they are contained in the half plane $\{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\}$.



Vector field $(\operatorname{Re}\sqrt{z}, \operatorname{Im}\sqrt{z})$ and colouring of the complex plane for $\sqrt{z}$.



Colouring and level curves of constant real (dotted) and imaginary (solid) part for $\sqrt{z}$; they intersect perpendicularly.



Colouring and level curves of constant modulus (dotted) and argument (solid) for $\sqrt{z}$; they intersect perpendicularly.

IV. Sequences and Series

IV.1. Sequences of Complex Functions

References: [DET, Section 4.2]

Recall that a sequence of functions $f_n : D \to \mathbb{C}$ *converges pointwise* to f on D if $\lim_{n\to\infty} f_n(z) = f(z)$ for each $z \in D$.

Definition. A sequence of functions $f_n : D \to \mathbb{C}$, $n \in \mathbb{N}$, *converges uniformly* to a function $f : D \to \mathbb{C}$, $f_n \to f$ on D, if

$$\forall \varepsilon > 0\, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N \text{ and } \forall z \in D : |f_n(z) - f(z)| < \varepsilon.$$

Remark. If $f_n \to f$ on D and $\tilde{D} \subset D$ then $f_n \to f$ on $\tilde{D}$.

(ex) Show: $f_n(z) = \frac{1}{z-n}$ converges pointwise but not uniformly on $\mathbb{C} \setminus \mathbb{N}$ to the zero function. However, the sequence f_n converges uniformly on $B_1(0)$ to the zero function.

Theorem IV.1.1 (Cauchy criterion for uniform convergence)**.** *A sequence of functions $f_n : D \to \mathbb{C}$, $n \in \mathbb{N}$, is uniformly convergent iff*

$$\forall \varepsilon > 0\, \exists N \in \mathbb{N} \text{ s.t. } \forall m, n \geq N \text{ and } \forall z \in D : |f_n(z) - f_m(z)| < \varepsilon.$$

Proof. ...as in M11 (ex). □

Remark. If you are familiar with metric spaces (see M41), then the previous statements can be reformulated as: The set of all holomorphic functions on D is a complete metric space where the metric d is given by the supremum metric $d(f, g) = \sup_{z\in D} |f(z) - g(z)|$. Indeed, the Cauchy criterion for uniform convergence is a statement about Cauchy sequences in this metric space!

Definition. We say that $f_n \to f$ *locally uniformly* on D if

$$\forall z \in D\ \exists \varrho > 0 : f_n|_{\overline{B_\varrho(z)}} \to f|_{\overline{B_\varrho(z)}}.$$

Remark. $f_n \to f$ locally uniformly on D iff $f_n \to f$ on every compact subset $K \subset D$. This is sometimes called "normal convergence".

(ex) Prove that $f_n \to f$ locally uniformly on D iff $f_n \to f$ on every compact subset $K \subset D$. *Hint:* Use the topological definition of compactness.

Theorem IV.1.2 (Continuity of locally uniform limits)**.** *If a sequence of continuous functions $f_n : D \to \mathbb{C}$ converges locally uniformly to $f : D \to \mathbb{C}$ on D then f is continuous.*

Proof. . . . as in M11. ☐

(ex) Prove that the limit function of a locally uniformly convergent sequence of continuous functions is continuous.

We will need the following technical lemma later (see proof of Lemma IV.6.1).

Lemma IV.1.3. *Let $g : D \to A$ be continuous and $f_n \to f$ locally uniformly on A. Then $f_n \circ g \to f \circ g$ locally uniformly on D.*

(ex) Prove Lemma IV.1.3.

Lemma IV.1.4. *If $\gamma : [a, b] \to \mathbb{C}$ is piecewise regular, $\mathit{\Gamma} = \gamma([a, b])$, and $f : \mathit{\Gamma} \to \mathbb{C}$ is the uniform limit of a sequence of continuous functions $f_n : \mathit{\Gamma} \to \mathbb{C}$ then*

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma \lim_{n\to\infty} f_n(z)\,\mathrm{d}z = \lim_{n\to\infty} \int_\gamma f_n(z)\,\mathrm{d}z.$$

Proof. Take $\varepsilon > 0$ and let $L = \int_a^b |\gamma'(t)|\,\mathrm{d}t > 0$ be the length of γ. Since $f_n \to f$ on $\mathit{\Gamma}$ there is an $N \in \mathbb{N}$ so that

$$|f_n(z) - f(z)| < \frac{\varepsilon}{L}$$

for all $n \geq N$ and $z \in \mathit{\Gamma}$. Thus

$$\left| \int_\gamma f_n(z)\,\mathrm{d}z - \int_\gamma f(z)\,\mathrm{d}z \right| \leq \max_{z\in\mathit{\Gamma}} |f_n(z) - f(z)|\,L < \varepsilon$$

or $n \geq N$ by the ML-inequality, that is, $\int_\gamma f_n(z)\,\mathrm{d}z \to \int_\gamma f(z)\,\mathrm{d}z$. ☐

Theorem IV.1.5 (Holomorphicity of locally uniform limits)**.** *If a sequence of holomorphic functions $f_n : D \to \mathbb{C}$ converges locally uniformly to $f : D \to \mathbb{C}$, then f is holomorphic and $f_n' \to f'$ locally uniformly.*

Proof. Fix $z_0 \in D$ and $\varrho > 0$ so that $f_n \to f$ on $\overline{B}_{2\varrho}(z_0) \subset D$. Then, using Cauchy's formula for the f_n's and Lemma IV.1.4,

$$\begin{aligned}
f(z) &= \lim_{n\to\infty} f_n(z)\\
&= \lim_{n\to\infty} \frac{1}{2\pi i} \int_{|w-z_0|=2\varrho} \frac{f_n(w)}{w-z}\,\mathrm{d}w\\
&= \frac{1}{2\pi i} \int_{|w-z_0|=2\varrho} \frac{\lim_{n\to\infty} f_n(w)}{w-z}\,\mathrm{d}w\\
&= \frac{1}{2\pi i} \int_{|w-z_0|=2\varrho} \frac{f(w)}{w-z}\,\mathrm{d}w
\end{aligned}$$

for $z \in B_{2\varrho}(z_0)$, which is differentiable at z, by Lemma III.2.6, with

$$f'(z) = \frac{1}{2\pi i} \int_{|w-z_0|=2\varrho} \frac{f(w)}{(w-z)^2}\,\mathrm{d}w.$$

A calculation as before shows ex that $f'(z) = \frac{1}{2\pi i}\int_{|w-z_0|=2\varrho} \frac{\lim\limits_{n\to\infty} f_n(w)}{(w-z)^2}\,\mathrm{d}w = \lim\limits_{n\to\infty} f_n'(z)$.

It remains to show that $f_n' \to f'$ locally uniformly.

Thus take $\varepsilon > 0$ and $N \in \mathbb{N}$ so that

$$|f_n(z) - f(z)| < \frac{\varrho}{2}\,\varepsilon$$

for all $z \in \varGamma = \{w \mid |w - z_0| = 2\varrho\}$ and $n \geq N$.

Then, whenever $z \in \overline{B}_\varrho(z_0)$,

$$\begin{aligned} |f_n'(z) - f'(z)| &= \frac{1}{2\pi}\left|\int_{|w-z_0|=2\varrho} \frac{f_n(w) - f(w)}{(w-z)^2}\,\mathrm{d}w\right| \\ &\leq \frac{4\pi\varrho}{2\pi}\max_w \frac{|f_n(w) - f(w)|}{|w-z|^2} \\ &\leq \frac{2\varrho}{\varrho^2}\max_w |f_n(w) - f(w)| \\ &< \varepsilon \end{aligned}$$

by the ML-inequality. Hence $f_n' \to f'$ on $\overline{B}_\varrho(z_0)$.

Thus $f_n' \to f'$ locally uniformly on D as $z_0 \in D$ was arbitrary. ☐

Remark. Compare this with the corresponding M11-theorem (interchanging differentiation and taking limits): there we needed to assume uniform convergence of the derivatives!

ex Let $f_n : B_1(0) \to \mathbb{C}$, $z \mapsto f_n(z) = \frac{z^{n+1}}{n+1}$. Prove that $f_n \to 0$ on $B_1(0)$ and $f_n' \to 0$ locally uniformly but not uniformly on $B_1(0)$.

IV.2. Power Series & the Cauchy-Taylor Theorem

References: [DET, Sections 4.3 & 4.4] and [ST, Sections 3.3 & 3.4 & 10.2] (Weierstrass M-test: [DET, Theorem 4.3.12]; Ratio and root test: [DET, Theorems 4.3.5 & 4.3.6]; Cauchy-Taylor Theorem: [DET, Theorem 4.4.2] and [ST, Theorem 10.3])

Recall (from M7)**.** An infinite series $\sum_{k=0}^{\infty} w_k$ is called *convergent* if the sequence $(\sum_{k=0}^{n} w_k)_{n\in\mathbb{N}}$ of partial sums is convergent.

If $\sum_{k=0}^{\infty} w_k$ is convergent then $w_k \to 0$ as $k \to \infty$.

If $\sum_{k=0}^{\infty} w_k$ is *absolutely convergent*, i.e., if the series $\sum_{k=0}^{\infty} |w_k|$ is convergent, then it is convergent.

Theorem IV.2.1 (Weierstrass M-test)**.** *If $f_k : D \to \mathbb{C}$, $k \in \mathbb{N}$, satisfy $|f_k(z)| \leq M_k$ for all $z \in D$ and $\sum_{k=0}^{\infty} M_k$ is convergent, then $\sum_{k=0}^{\infty} f_k$ converges absolutely and uniformly on D.*

Proof. …as in M11. ☐

(ex) Let $f_k : D \to \mathbb{C}$, $k \in \mathbb{N}$ and $D \subset \mathbb{C}$ *any* subset of $\mathbb{C}$, satisfy $|f_k(z)| \leq M_k$ for all $k \in \mathbb{N}$ and $z \in D$ and assume that $\sum_{k=0}^{\infty} M_k$ converges. Prove that $\sum_{k=0}^{\infty} f_k$ converges absolutely and uniformly on D (i.e., the Weierstrass M-test).
Hint: Use the Cauchy criterion.

Recall (from M7)**.** A *power series* is a series of the form

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

Given a power series there is a number $R \in [0, \infty]$ so that the series

(i) converges absolutely for $|z - z_0| < R$ and
(ii) diverges for $|z - z_0| > R$.

To compute R one can, for example, use the *root test*[1]:

$$R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}}.$$

(Note: Another way to test for absolute convergence, is the *ratio test*[2]; if $\lim_{k\to\infty} |a_{k+1}/a_k|$ exists, then $R = 1/\lim_{k\to\infty} |a_{k+1}/a_k|$.)

Theorem IV.2.2. *Functions defined by power series are holomorphic; the derivative is obtained by differentiating term by term.*

Proof. (This is similar to the proof of a corresponding statement in M11.)
By the Weierstrass M-test, the power series converges uniformly on every $B_r(z_0)$, where $r < R$ is smaller than the radius of convergence. Thus the power series converges locally uniformly and the claim follows from Theorem IV.1.5. □

Theorem IV.2.3 (Cauchy-Taylor Theorem)**.** *Let $f : D \to \mathbb{C}$ be holomorphic and $B_R(z_0) \subset D$ for some $z_0 \in D$ and $R \in (0, \infty]$.*
Then, for $z \in B_R(z_0)$,

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad \text{with} \qquad a_k = \frac{f^{(k)}(z_0)}{k!}$$

Proof. Suppose, w.l.o.g., $z_0 = 0$ and take $z \in B_R(0)$; now choose r so that $|z| < r < R$.

---

[1] Let $M = \limsup_{k\to\infty} |w_k|^{1/n}$. If $M < 1$, the series $\sum_{k=1}^{\infty} w_k$ converges; if $M > 1$, the series $\sum_{k=1}^{\infty} w_k$ diverges; if $M = 1$, then no conclusion can be reached about the convergence of $\sum_{k=1}^{\infty} w_k$.

[2] Let $M = \limsup_{k\to\infty} |w_{k+1}/w_k|$ and $m = \liminf_{k\to\infty} |w_{k+1}/w_k|$. Then, if $M < 1$, the series $\sum_{k=1}^{\infty} w_k$ converges; if $m > 1$, the series $\sum_{k=1}^{\infty} w_k$ diverges; if $m \leq 1 \leq M$, then no conclusion can be reached about the convergence of $\sum_{k=1}^{\infty} w_k$.

Then

$$\begin{aligned}
2\pi i\, f(z) &= \int_{|w|=r} \frac{f(w)}{w-z}\,\mathrm{d}w \\
&= \int_{|w|=r} \frac{f(w)}{w}\,\frac{1}{1-\frac{z}{w}}\,\mathrm{d}w \\
&= \int_{|w|=r} \frac{f(w)}{w} \sum_{k=0}^{\infty} \left(\frac{z}{w}\right)^k \mathrm{d}w \\
&\overset{(\star)}{=} \sum_{k=0}^{\infty} \left(\int_{|w|=r} \frac{f(w)}{w^{k+1}}\,\mathrm{d}w \right) z^k \\
&= 2\pi i \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, z^k
\end{aligned}$$

by Cauchy's formula and Cauchy's formulas for the derivatives.

The interchange of summation and integration in ($\star$) is justified by the uniform convergence of

$$w \mapsto \sum_{k=0}^{\infty} \frac{f(w)}{w} \left(\frac{z}{w}\right)^k = \frac{f(w)}{w} \sum_{k=0}^{\infty} \left(\frac{z}{w}\right)^k$$

for $\left|\frac{z}{w}\right| \equiv \frac{|z|}{r} < 1$ (see Lemma IV.1.4). □

Remark. In the course of the proof we learnt that

(i) the coefficients in the Taylor series expansion (around z_0) are equivalently given by

$$a_k = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{k+1}}\,\mathrm{d}w.$$

(ii) the radius of convergence R of the Taylor series is at least the radius of the largest disk that is entirely contained in the domain D of f: if $B_\varrho(z_0) \subset D$, then $R \geq \varrho$.

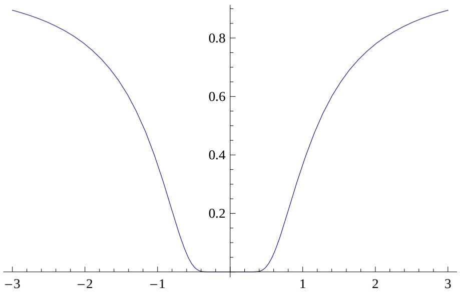
(ex) Let f be an entire function and z_0 arbitrary. Show that the Taylor series expansion of f at z_0 has radius of convergence $R = \infty$.

Definition. We say that a function $f : D \to \mathbb{C}$ is *analytic* if

$$\forall z_0 \in D\, \exists \varrho > 0 \text{ s.t. } \forall z \in B_\varrho(z_0) : f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

for some sequence $(a_k)_{k \in \mathbb{N} \cup \{0\}} \subset \mathbb{C}$.

Remark. The above theorems say: holomorphic and analytic functions are the same (see Theorem IV.4.1).
In $\mathbb{R}$ not even C^∞-functions need to be analytic, e.g., for $f(x) = \mathrm{e}^{-1/x^2}$ one has $f^{(k)}(0) = 0$ (f is depicted on the right).



Example.

$$\begin{aligned}
\mathrm{e}^z &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \\
\sin z &= \frac{\mathrm{e}^{i\,z} - \mathrm{e}^{-i\,z}}{2i} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2\,k+1}}{(2\,k+1)!} \\
\cos z &= \frac{\mathrm{e}^{i\,z} + \mathrm{e}^{-i\,z}}{2} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2\,k}}{(2\,k)!}
\end{aligned}$$

IV.3. The Identity Theorem and the Maximum Principle

REFERENCES: [DET, Section 4.4] and [ST, Sections 10.5 & 10.8]
(Identity Theorem for holomorphic functions: [DET, Theorem 4.4.4] and [ST, Proposition 10.10]; Maximum Modulus Theorem: [DET,] and [ST, Theorem 10.14])

Theorem IV.3.1 (Identity Theorem for power series)**.** *Let*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

be a power series with radius of convergence $R > 0$*. Suppose* $f(z_n) = 0$ *for a sequence* $(z_n)_{n\in\mathbb{N}} \subset B_R^*(z_0) = B_R(z_0) \setminus \{z_0\}$ *with* $z_n \to z_0$*. Then* $a_k = 0$ *for all* $k = 0, 1, \dots,$ *that is,* $f \equiv 0$*.*

Proof. …by induction:

- $k = 0$: First note that, since $z \mapsto f(z)$ is continuous,

  $$a_0 = f(z_0) = \lim_{n\to\infty} f(z_n) = 0.$$

- $k - 1 \to k$: Now suppose $a_0 = \dots = a_{k-1} = 0$ and consider

  $$g(z) = \frac{f(z)}{(z - z_0)^k} = \sum_{j=0}^{\infty} a_{k+j} (z - z_0)^j.$$

  This is a power series with radius of convergence R:

  - if $0 < |z - z_0| < R$ then $\sum_{j=0}^{\infty} a_{k+j}(z - z_0)^j = \frac{f(z)}{(z-z_0)^k}$ converges (absolutely) with $f(z)$ and
  - if $|z - z_0| > R$ then $\sum_{j=0}^{\infty} a_{k+j}(z - z_0)^j = \frac{f(z)}{(z-z_0)^k}$ diverges.

  Thus, as a power series with radius of convergence R the function g is continuous at $z = z_0$, so that

  $$a_k = g(z_0) = \lim_{n\to\infty} \frac{f(z_n)}{(z_n - z_0)^k} = 0$$

  since $z_n \neq z_0$ for all $n \in \mathbb{N}$.

□

Remark. Thus a power series expansion of a function is unique: if

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = \sum_{k=0}^{\infty} b_k (z - z_0)^k$$

on $B_R(z_0)$ for some $R > 0$, then $a_k = b_k$ for all $k \in \mathbb{N} \cup \{0\}$.

ex Find the Taylor series expansion of $z \mapsto f(z) = \frac{1}{1+z^2}$ with $z_0 = 0$ and determine its radius of convergence.
Hint: Do not compute derivatives.

Theorem IV.3.2 (Identity Theorem for holomorphic functions)**.** *Suppose a holomorphic function $f : D \to \mathbb{C}$ on a domain D satisfies $f(z_n) = 0$ for a sequence $(z_n)_{n\in\mathbb{N}} \subset D \setminus \{z_0\}$ with $z_n \to z_0 \in D$. Then $f \equiv 0$.*

Proof. Let $N = \{z \in D \,|\, f(z) = 0\}$ and

$$N' = \{z \in D \,|\, \exists (z_n)_{n\in\mathbb{N}} \subset N \setminus \{z\} \text{ s.t. } z = \lim_{n\to\infty} z_n\}.$$

(N' is the set of *limit* (or *accumulation*) *points* of N).

As f is continuous we have $N' \subset N$, i.e., $f(z) = 0$ for all $z \in N'$ (in topological terms, this is simply the statement that N is a closed set).

By definition $z_0 \in N'$, so that $N' \neq \varnothing$.

Now $N' \subset D$ is closed in D: take $z \in D \setminus N'$ and suppose

$$\forall n \in \mathbb{N} : B_{\frac{1}{n}}(z) \cap N' \neq \varnothing,$$

i.e., we have $z_n \in N' \subset N$ with $|z_n - z| < \frac{1}{n}$ – this contradicts the assumption $z \notin N'$ since $z_n \to z$ as $n \to \infty$. So, N' contains all its limit points and is therefore closed.

Also, $N' \subset D$ is open in D: by the Cauchy-Taylor Theorem

$$\forall z_0 \in N' \subset D \;\exists \varrho > 0 \text{ s.t. } \forall z \in B_\varrho(z_0) \text{ we have } f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

and, by Theorem IV.3.1, $f \equiv 0$ on $B_\varrho(z_0)$.

Hence, by Lemma I.5.2, $N' = D$ (wherefore $f \equiv 0$ on all of D). ☐

Remark. Alternative proof (Not examinable!)

> *Proof.* Let $\tilde{z} \in D$ and choose some path $\gamma : [0, 1] \to D$ (D is connected!) with $\gamma(0) = z_0$ and $\gamma(1) = \tilde{z}$; w.l.o.g., we may assume that γ is simple. Now let
>
> $$T = \{t \in [0, 1] \,|\, \forall s \in [0, t] : f(\gamma(s)) = 0\}.$$
>
> Since f is continuous at z_0
>
> $$f(z_0) = \lim_{n\to\infty} f(z_n) = 0.$$
>
> Hence $T \neq \varnothing$ since $0 \in T$ and T is bounded above (by 1) so that
>
> $$\tau = \sup T \in [0, 1].$$

Note that, if $0 \leq t < \tau$, then there is $\tilde{t} \in [t, \tau)$ so that $f(\gamma(s)) = 0$ for all $s \in [0, \tilde{t}]$. Consequently, $t \in T$.

By the Cauchy-Taylor Theorem

$$\exists \varrho > 0 \, \forall z \in B_\varrho(z_0) : f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

and by the Identity Theorem for power series $f \equiv 0$ on $B_\varrho(z_0)$ since $f(z_n) = 0$ for a sequence with $z_n \to z_0$, $z_n \neq z_0$. And, since γ is continuous,

$$\exists \delta > 0 \, \forall t < \delta : \gamma(t) \in B_\varrho(z_0);$$

hence $[0, \delta) \subset T$ and $\tau \geq \delta > 0$.

Now suppose, for a contradiction, $\tau < 1$ and denote $w_0 = \gamma(\tau)$. Then, by the Cauchy-Taylor Theorem again,

$$\exists \varrho > 0 \, \forall z \in B_\varrho(z_0) : f(z) = \sum_{k=0}^{\infty} a_k (z - w_0)^k$$

and $w_n = \gamma((1 - \frac{1}{n})\tau)$ defines a sequence with $f(w_n) = 0$ and $w_n \to w_0$, where $w_n \neq w_0$ since γ is simple. Hence, by the Identity Theorem for power series, $f \equiv 0$ on $B_\varrho(z_0)$. But, since γ is continuous,

$$\exists \delta > 0 \, \forall t \in (\tau - \delta, \tau + \delta) : \gamma(t) \in B_\varrho(z_0);$$

in particular, $\tau + \frac{\delta}{2} \in T$, contradicting the definition of $\tau = \sup T$.

Consequently, we have $\tau = 1$ and $f(\tilde{z}) = f(z)$. □

Remark. (i) Thus if two holomorphic functions $f, g : D \to \mathbb{C}$ agree on a set $A \subset D$ that has an limit/accumulation point, then they agree on D.

(ii) If $f(z) = g(z)$ for $z \in B_\varrho(z_0)$, then $f(z) = g(z)$ for all $z \in D$ ("if two functions agree on a disk, they agree everywhere").

(iii) If $\gamma : [a, b] \to D$ is a non-constant path and $f \circ \gamma = g \circ \gamma$ then $f(z) = g(z)$ for all $z \in D$ ("if two functions agree on a path, they agree everywhere").

(iv) If f and g are entire and $f(x) = g(x)$ for all $x \in \mathbb{R}$ then $f = g$; in particular, $\exp, \cos, \sin : \mathbb{C} \to \mathbb{C}$ are uniquely defined by their values on real arguments ("if two functions agree on the real line, they agree everywhere").

(v) These are all instances of the statement: The extension of a function to a larger domain is unique, see Section IV.5. [3].

ex Let $B_1(0)$ be the open unit disc. Can you find a nonzero analytic function on $B_1(0)$ that has infinitely many zeros in $B_1(0)$? If yes, give your example and prove your claim; if not, give your reasoning why not.

---

[3] A function $f : D \to \mathbb{C}$ is said to be an *extension function* of $h : S \to \mathbb{C}$ if $S \subset D$ and $f(z) = h(z)$ for all $z \in S$.

Suppose that D is a domain and S is a subset with a limit point in D. Then, the Identity Theorem shows that if a function $h : S \to \mathbb{C}$ has an extension $f : D \to \mathbb{C}$ which is holomorphic, this extension is unique.

This also means that the Taylor expansion of f about any point in its domain contains all the information required to determine f throughout the domain. This observation leads to the important topic of *analytic continuations*, see Section IV.5.

(ex) Show (without calculations): Given the complex sine and cosine, we have $\cos^2 z + \sin^2 z = 1$.
Also argue that the compound angle formulae (e.g., $\cos(z+w) = \cos z \cos w - \sin z \sin w$ with $z, w \in \mathbb{C}$) hold for the complex sine and cosine.

Theorem IV.3.3 (The Maximum Modulus Theorem)**.** *Let $f : D \to \mathbb{C}$ be holomorphic in a domain and suppose that $|f| : D \to \mathbb{R}$ has a maximum at $z_0 \in D$. Then f is constant.*

Proof. Since D is open there is $\varrho > 0$ so that $B_\varrho(z_0) \subset D$. Then, by the Local Maximum Modulus Theorem (Theorem III.4.4), $f \equiv c \in \mathbb{C}$ is constant on $B_\varrho(z_0)$.

And, by the Identity Theorem $f \equiv c$ in D since $g(z) = c$ defines a holomorphic function on D with $f(z) = g(z)$ for $z \in B_\varrho(z_0)$. ☐

(ex) Suppose that $x^2 + y^2 \leq 1$. Prove that $(x^2 - y^2 - 1)^2 + 4x^2y^2$ attains its maximum value when $x = 0$, $y = \pm 1$.

(ex) In this exercise we show that there is **no** holomorphic function $f : B_1(0) \to B_1(0)$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{3}{5}$.

(i) Consider the two Möbius transformations φ and ψ given by

$$\varphi(z) = \frac{z + \frac{1}{2}}{1 + \frac{1}{2}z} \qquad \text{and} \qquad \psi(z) = \frac{z - \frac{3}{4}}{1 - \frac{3}{4}z}.$$

- Show: φ and ψ are holomorphic in $B_1(0)$ (and continuous on $\partial B_1(0)$).
- Show: $\varphi(\partial B_1(0)) = \partial B_1(0)$ and $\psi(\partial B_1(0)) = \partial B_1(0)$, i.e., φ and ψ map the unit circle on the unit circle.
  Hint: Note that three distinct (noncollinear) points determine a circle.
- Conclude that φ and ψ map $B_1(0)$ into itself.
  Hint: Maximum Modulus Theorem.

(ii) Prove the so-called *Schwarz' Lemma*[4]:
Suppose $\Phi : B_1(0) \to B_1(0)$ with $\Phi(0) = 0$ (i.e., Φ maps the unit disk into the unit disk and the origin to the origin), then $|\Phi(z)| \leq |z|$ for all $z \in B_1(0)$ and $|\Phi'(0)| \leq 1$. Furthermore, if $|\Phi'(0)| = 1$ or $|\Phi(z)| = |z|$ for some $z \in B_1^*(0)$, then Φ is a rotation: $\Phi(z) = \mathrm{e}^{i\theta} \cdot z$ for some real constant θ.

Hint: Apply the Maximum Modulus Theorem to the function

$$g(z) = \begin{cases} \Phi(z)/z & \text{if } z \in B_1^*(0) \\ \Phi'(0) & \text{if } z = 0. \end{cases}$$

Here, $B_1^*(0) = B_1(0) \setminus \{0\}$ denotes the punctured unit disk.

(iii) Now show that there is no holomorphic function $f : B_1(0) \to B_1(0)$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{3}{5}$.
Hint: Suppose such a function exists and consider $\Phi = \psi \circ f \circ \varphi$.

---

[4] Not to be confused with the earlier statement by the same name that the second mixed partial derivatives commute.

IV.4. Characterisation of Holomorphicity

We summarise the characterisation of holomorphic functions we have obtained:

Theorem IV.4.1. *For a continuous function f on a domain D the following statements are equivalent:*

(i) *f is holomophic in D.*

(ii) *$f(x+iy) = u(x,y) + iv(x,y)$ with C^1-functions $u, v : D \to \mathbb{R}$ which satisfy the Cauchy-Riemann equations $u_x = v_y$ and $v_x = -u_y$.*

(iii) *$f(z) = F(z, \overline{z})$ where F is a holomorphic function of two variables such that $F_2 = 0$ (i.e., $\frac{\partial f}{\partial \overline{z}}(z) = 0$).*

(iv) *$\int_\Gamma f \,\mathrm{d}z = 0$ for every simple closed contour Γ with $I_\Gamma \subset D$.*

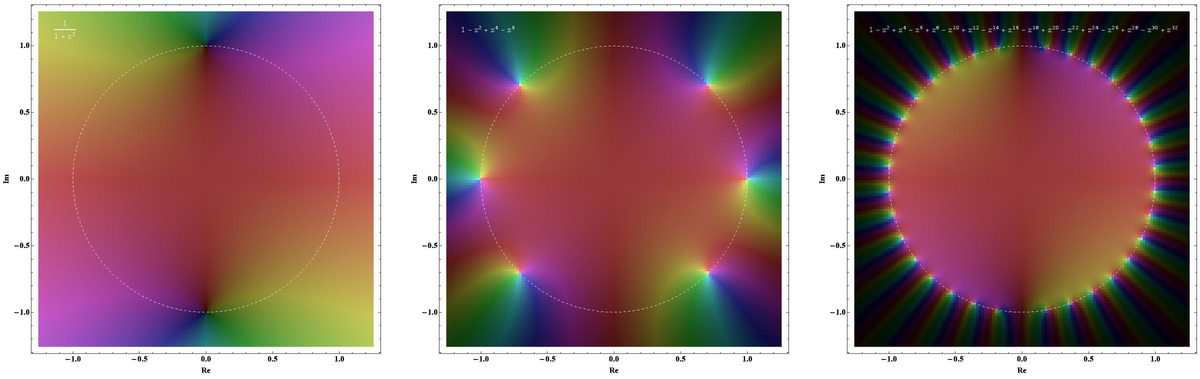
(v) *f is analytic in D.*

Proof. (i)⇒(ii): necessary Cauchy-Riemann conditions (Theorem II.3.1) and existence of continuous f' (Corollary III.3.3).
(ii)⇒(i): sufficient Cauchy-Riemann conditions (Theorem II.3.5).
(ii)⇔(iii): Theorem II.5.1.
(i)⇒(iv): Cauchy's Theorem (Theorem III.2.2).
(iv)⇒(i): Morera's Theorem (Corollary III.3.4).
(i)⇒(v): Cauchy-Taylor Theorem (Theorem IV.2.3).
(v)⇒(i): Theorem IV.2.2. ☐

IV.5. Analytic Continuation & Riemann's Zeta Function (Not examinable!)

References: [DET, Section 4.5] and [ST, Section 14.3]

Consider the exercise on p. 69 again: One can check that the Taylor series expansion of $z \mapsto f(z) = \frac{1}{1+z^2}$ with $z_0 = 0$ has radius of convergence $R = 1$. So, the function defined by this Taylor series(!) is certainly not defined for $|z| > 1$.



The function $f(z) = \frac{1}{1+z^2}$ on the left, and two partial sums of its Taylor series around 0, namely up to order 6 in the middle and up to order 32 on the right. Observe that the Taylor series converges in the unit disk to f, but not outside.

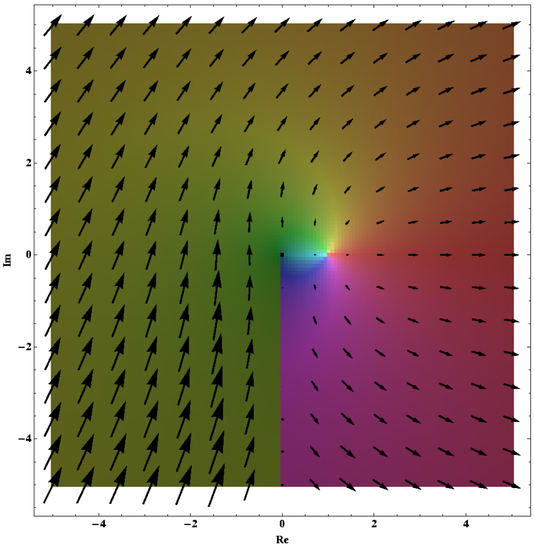
We now turn this observation around and ask: Given a holomorphic function on some "small domain", can we find a holomorphic function (and if so, how many[5] such functions) on a "bigger domain" (or at least a domain that has some overlap with the former domain) that extends the function in question.

Definition. If f_1 is holomorphic on a domain D_1 and f_2 is holomorphic on a domain D_2, where $D_1 \cap D_2 \neq \varnothing$ and $f_1(z) = f_2(z)$ for all $z \in D_1 \cap D_2$, then we say that f_2 is a *direct analytic continuation* of f_1 to the domain D_2.

The identity theorem has the following consequences (see [DET, Theorem 4.5.1 & Corollary 4.5.1] for a proof):

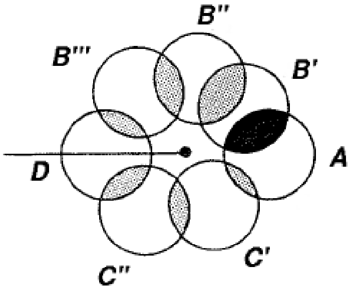
Theorem IV.5.1. *If f_2 is holomorphic on a domain D_2 and f_3 is holomorphic on D_3 and both functions are direct analytic continuations of a holomorphic function f_1 on a domain D_1 such that $D_2 \cap D_3$ is connected and not empty and $D_1 \cap D_2 \cap D_3 \neq \varnothing$, then $f_2 = f_3$ in $D_2 \cap D_3$.*
In particular, if f_1 is holomorphic on D_1 and D_2 is domain such that $D_1 \cap D_2 \neq \varnothing$, then a direct analytic continuation of f_1 into D_2 is unique if it exists. ☐

The complex logarithm provides a good example that all assumptions in the previous theorem are necessary: For example, the principal value of the logarithm on p. 61 (i.e., on the cut plane $D_2 = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$) and the logarithm on the cut plane $D_3 = \mathbb{C} \setminus i \cdot \mathbb{R}_{\leq 0}$ (see right) are direct analytic continuations of the principal value of the logarithm restricted to the domain $D_1 = \{z \,|\, \operatorname{Re} z > 0\}$, however $D_2 \cap D_3$ is not connected and indeed these two analytic continuations differ in the lower left quadrant.



Vector field and colouring of $\mathbb{C}$ for a logarithm on the cut plane $\mathbb{C} \setminus i \cdot \mathbb{R}_{\leq 0} = \{z \,|\, \arg(z) \neq -\frac{\pi}{2}\}$.

Furthermore, the complex logarithm also shows that by considering different sequences of (unique!) direct analytic continuations, one might end up with different functions: We begin with the principal value of the logarithm restricted to the disk A, then we can find a unique direct analytic continuation to the disk B'. But this function has a direct analytic continuation to the disk B'', and that function one to the disk B''', which in turn has one to the disk D. But what happens if we consider the sequence of disks A, C', C'' to arrive in D? We end up with a different function, namely[6] we get different branches of the logarithm and the two functions on D differ by a constant of $2\pi i$ (the figure to the right is adapted from [ST, Fig. 14.9]).



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[5] In real analysis, such a question would "not make sense": there are many ways to extend a (real) differentiable function (even if it is C^∞)!

In 8 steps to Riemann's Zeta Function[7]

Riemann's Zeta function is defined by an analytic continuation. One way to establish the Riemann's Zeta function on all of $\mathbb{C} \setminus \{1\}$ is by considering (and proving) the following steps:

(1) We begin with the function defined by an infinite series:

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}.$$

This series is convergent iff $\operatorname{Re} z > 1$ (and we always use the principal value $n^{-z} = \mathrm{e}^{-z \operatorname{Log}(n)}$ here).

(2) We consider the function ζ_1 defined by

$$\zeta_1(z) = z \int_1^{\infty} \frac{[t] - 1}{t^{z+1}} \, \mathrm{d}t + \frac{z}{z-1}$$

(where $[t]$ denotes the integer part of the real number t). We can show that ζ_1 is holomorphic on $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0, z \neq 1\}$ and that $\zeta_1 = \zeta$ for $\operatorname{Re} z > 1$. Thus, ζ_1 is a direct analytic continuation of ζ.

(3) We consider the function ζ_2 defined by

$$\zeta_2(z) = z \int_1^{\infty} \frac{[t] - t + \frac{1}{2}}{t^{z+1}} \, \mathrm{d}t + \frac{1}{z-1} + \frac{1}{2}.$$

One can show that $\zeta_2 = \zeta_1$ on $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0, z \neq 1\}$.

(4) One can show that ζ_2 is holomorphic on $\{z \in \mathbb{C} \mid \operatorname{Re} z > -1, z \neq 1\}$ and thus ζ_2 is a direct analytic continuation of ζ_1 (and ζ).

(5) We consider the function ζ_3 defined by

$$\zeta_3(z) = z \int_0^{\infty} \frac{[t] - t + \frac{1}{2}}{t^{z+1}} \, \mathrm{d}t.$$

One can show that ζ_3 is holomorphic *and* $\zeta_3 = \zeta_2$ on $\{z \in \mathbb{C} \mid -1 < \operatorname{Re} z < 0\}$. Hence, ζ_3 is an analytic but not a direct analytic contnuation of ζ.

(6) One establishes that

$$\zeta_3(z) = -2^z \, \pi^{z-1} \, z \sin\left(\frac{1}{2}\pi \, z\right) \, \Gamma(-z) \, \zeta(1-z)$$

if $-1 < \operatorname{Re} z < 0$. Here, Γ denotes the (complex) gamma function (here is another analytic continuation hidden!).

---

[6] In the Riemann surface picture, we "go up the helix" if we go round the origin counter-clockwise and "down the helix" if we go round the origin in clockwise direction, compare p. 60.

[7] This is adapted from Open University Complex Analysis Course Team: "Course M332 Unit 15, Complex Analysis: Number Theory", The Open University Press, Milton Keynes (1975); library: `513.317 OPE`.

(7) One shows that the function

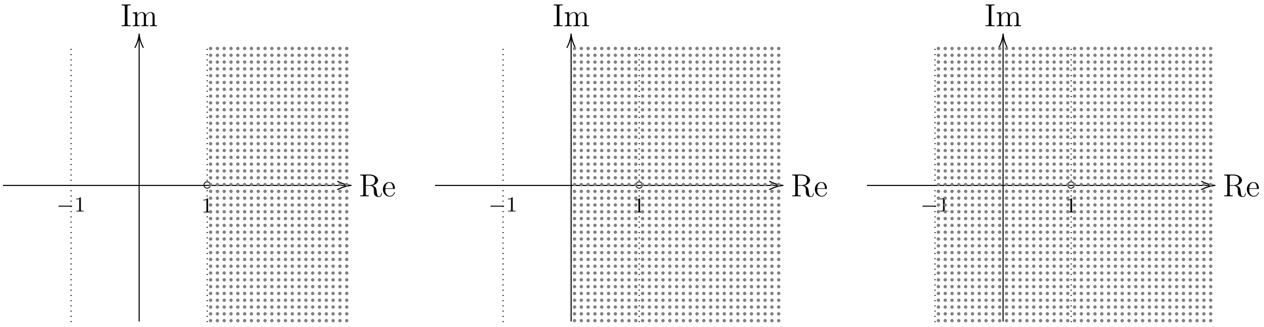
$$\zeta_4(z) = -2^z\,\pi^{z-1}\,z\,\sin\left(\frac{1}{2}\pi\,z\right)\,\Gamma(-z)\,\zeta(1-z)$$

is holomorphic on the half-plane $\operatorname{Re} z < 0$ and so ζ_4 is a direct analytic continuation of ζ_3. By step (4), it is also a direct analytic continuation of ζ_2. It follows that Riemann's Zeta Function has been continued analytically onto $\mathbb{C} \setminus \{1\}$.

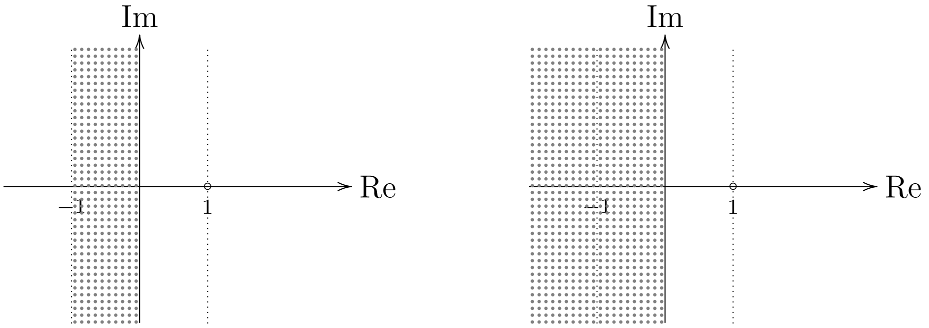
(8) By the "Permanence of Functional Relationships" we have

$$\zeta(z) = -2^z\,\pi^{z-1}\,z\,\sin\left(\frac{1}{2}\pi\,z\right)\,\Gamma(-z)\,\zeta(1-z)$$

where ζ now denotes the analytic continuation of ζ in step (1).



The domain on which ζ (see step (1)) is defined on the left, the domain on which ζ_1 (see step (2)) is holomorphic in the middle, and the domain on which ζ_2 (see step (4)) is holomorphic on the right.



The domain on which ζ_3 (see step (5)) is holomorphic on the left and the domain on which ζ_4 (see step (7)) is holomorphic on the right.

IV.6. Laurent Series

References: [DET, Section 4.6] and [ST, Section 11.1]
(Laurent's Theorem: [DET, Theorem 4.6.1] and [ST, Theorem 11.1])

The Taylor series for $\cos z$ is $1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$; it converges for every z.

So, the series $\frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \dots$ converges to $(\cos z)/z$ for every nonzero z; we have a series expansion for this function, and the fact that 0 is a "singularity" of the function explains, or is explained by, the presence of the term $1/z$ in the series.

Definition. A *Laurent series* is a series of the form

$$\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} + \sum_{k=0}^{\infty} a_k(z-z_0)^k;$$

we say that this series converges to $l = l_1 + l_2$ if

$$\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} \to l_1 \qquad \text{and} \qquad \sum_{k=0}^{\infty} a_k(z-z_0)^k \to l_2.$$

In this case we write

$$l = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k.$$

We call $\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k}$ the *principal part* (or *singular part*) and $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ the *regular part* (or *power series part*) of the Laurent series.

Remark. Given a series $\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k}$ there is an $R \in [0,\infty]$ so that the series

(i) diverges for $|z-z_0| < R$ and

(ii) converges (absolutely) for $|z-z_0| > R$.

Proof. W.l.o.g. $z_0 = 0$. Let $\varrho \in [0,\infty]$ denote the radius of convergence of the power series

$$\sum_{k=1}^{\infty} a_{-k} w^k.$$

Then, with $w = \frac{1}{z}$ and $R = \frac{1}{\varrho}$, the series

(i) converges (absolutely) when

$$|w| = \frac{1}{|z|} < \varrho \qquad \Leftrightarrow \qquad |z| > \frac{1}{\varrho} = R;$$

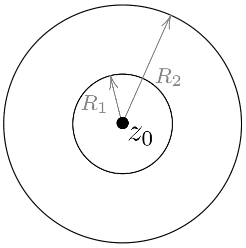
(ii) diverges whenever

$$|w| = \frac{1}{|z|} > \varrho \qquad \Leftrightarrow \qquad |z| < \frac{1}{\varrho} = R.$$

□

Thus a Laurent series converges in the intersection of the complement of a closed disc $\{z \in \mathbb{C} \mid |z-z_0| > R_1\}$ and an open disk $B_{R_2}(z_0)$, that is, in an *annulus*

$$A = \{z \in \mathbb{C} \mid R_1 < |z-z_0| < R_2\}.$$



Example. The annulus of convergence of the Laurent series

$$\sum_{k=-\infty}^{\infty} z^k$$

is empty: its regular part $\sum_{k=0}^{\infty} z^k$ has radius of convergence $R_2 = 1$ and the singular part $\sum_{k=1}^{\infty} z^{-k}$ converges for $|z| > R_1 = 1$.

Example. We determine the Laurent series expansions of $f(z) = \frac{1}{z^2-1}$.
First note that f is defined for $z \neq \pm 1$. Thus we determine Laurent series expansions for $z \in B_1(0)$ and for $z \in \{w \mid 1 < |w|\}$.

- $|z| < 1$: Here we have

  $$f(z) = -\frac{1}{1-z^2} = -\sum_{k=0}^{\infty} (z^2)^k = -\sum_{k=0}^{\infty} z^{2k},$$

  a power series, which is not surprising since f is holomorphic in $B_1(0)$.
- $1 < |z|$: In this case $\left|\frac{1}{z}\right| < 1$ and we have

  $$f(z) = \frac{1}{z^2}\frac{1}{1-\frac{1}{z^2}} = \frac{1}{z^2}\sum_{k=0}^{\infty}\left(\frac{1}{z^2}\right)^k = \sum_{k=-\infty}^{-1} z^{2k}$$

  since the geometric series $\sum_{k=0}^{\infty} w^k$ converges for $w = \frac{1}{z^2} \in B_1(0)$.

(ex) Find the Laurent series expansions of $f(z) = \frac{1}{z(1-z)(2-z)}$ for the annuli

$$\text{(i)} \quad 0 < |z| < 1, \qquad \text{(ii)} \quad 1 < |z| < 2, \qquad \text{(iii)} \quad 2 < |z|.$$

Hint: Do not compute integrals.

(ex) Find the Laurent series expansions of $f(z) = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{2-z}$ for the annuli

$$\text{(i)} \quad 0 < |z| < 1, \qquad \text{(ii)} \quad 0 < |z-1| < 1, \qquad \text{(iii)} \quad 0 < |z-2| < 1.$$

Hint: Do not compute integrals.

Lemma IV.6.1. *Given a Laurent series* $\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$, *there are numbers* $R_1, R_2 \in [0,\infty]$ *so that the series converges absolutely and locally uniformly in the annulus*

$$A = \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}.$$

Proof. W.l.o.g. $z_0 = 0$; we write

$$f_1(z) = \sum_{k=1}^{\infty} a_{-k} z^{-k} \qquad \text{and} \qquad f_2(z) = \sum_{k=0}^{\infty} a_k z^k.$$

We already know (from the Weierstrass M-test, as in the proof of Theorem IV.2.2) that $f_2(z)$ converges absolutely and locally uniformly on $B_{R_2}(0)$ for some $R_2 \in [0,\infty]$.

Similarly, we know that there is an $R_1 \in [0,\infty]$ so that the power series

$$f_1\left(\frac{1}{w}\right) = \sum_{k=1}^{\infty} a_{-k} w^k$$

converges absolutely and locally uniformly on $B_{\frac{1}{R_1}}(0)$. Moreover, since

$$\mathbb{C}\setminus\{0\} \ni z \mapsto w = \frac{1}{z} \in \mathbb{C}\setminus\{0\}$$

is continuous, $f_1(z)$ also converges locally uniformly for $|z| > R_1$ by Lemma IV.1.3.

Consequently the Laurent series $f_1(z) + f_2(z)$ converges absolutely and locally uniformly when $|z| > R_1$ *and* $|z| < R_2$. □

As a consequence we have a theorem similar to Theorem IV.2.2 for Laurent series:

Theorem IV.6.2. *Functions defined by Laurent series are holomorphic in their annuli of convergence; the derivative is obtained by differentiating term-by-term.*

Proof. This is a consequence of Lemma IV.6.1 and Theorem IV.1.5. □

Theorem IV.6.3 (Laurent's Theorem)**.** *Let $f : D \to \mathbb{C}$ be holomorphic and suppose that for some $z_0 \in \mathbb{C}$ and $R_1, R_2 \in [0,\infty]$ with $R_1 < R_2$*

$$A = \{z \in \mathbb{C} \,|\, R_1 < |z - z_0| < R_2\} \subset D.$$

Then, for $z \in A$,

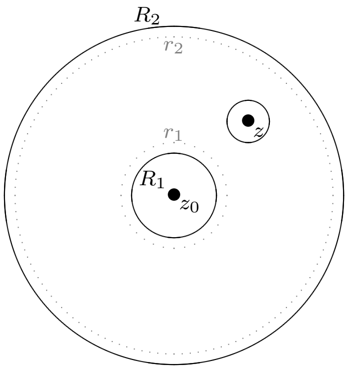
$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \qquad \text{with} \qquad a_k = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)\,\mathrm{d}w}{(w - z_0)^{k+1}}$$

for $k \in \mathbb{Z}$, where $R_1 < r < R_2$; this representation is unique.

Proof. W.l.o.g. $z_0 = 0$. Fix $z \in A$. Take r_1, r_2 so that

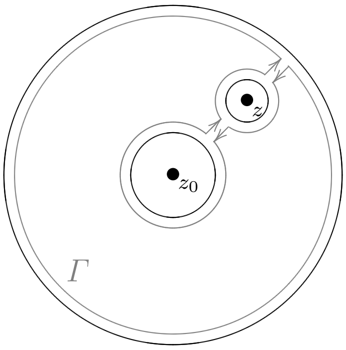
$$R_1 < r_1 < |z - z_0| = |z| < r_2 < R_2$$

and $\varrho > 0$ so that $B_{2\varrho}(z) \subset \{w \in \mathbb{C} \,|\, r_1 < |w| < r_2\}$ (it is actually enough to have $B_\varrho(z) \subset \{w \in \mathbb{C} \,|\, r_1 < |w| < r_2\}$).



By Cauchy's formula and the Homotopy version of Cauchy's theorem[8]

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{|w-z|=\varrho} \frac{f(w)}{w-z}\,\mathrm{d}w \\
&= \frac{-1}{2\pi i}\int_{|w|=r_1} \frac{f(w)}{w-z}\,\mathrm{d}w + \frac{1}{2\pi i}\int_{|w|=r_2} \frac{f(w)}{w-z}\,\mathrm{d}w \\
&= f_1(z) + f_2(z).
\end{aligned}
$$



From the proof of the Cauchy-Taylor Theorem we know

$$f_2(z) = \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i}\int_{|w|=r_2} \frac{f(w)}{w^{k+1}}\,\mathrm{d}w \right) z^k.$$

A similar computation for $f_1(z)$ yields

$$
\begin{aligned}
2\pi i\, f_1(z) &= -\int_{|w|=r_1} \tfrac{f(w)}{w-z}\,\mathrm{d}w \;=\; \int_{|w|=r_1} \tfrac{f(w)}{z}\tfrac{1}{1-\frac{w}{z}}\,\mathrm{d}w \\
&= \int_{|w|=r_1} \tfrac{f(w)}{z} \sum_{k=0}^{\infty} \left(\tfrac{w}{z}\right)^k \mathrm{d}w \\
&= \sum_{k=0}^{\infty} \int_{|w|=r_1} f(w)\, w^k\,\mathrm{d}w\, \tfrac{1}{z^{k+1}} \\
&= \sum_{k=1}^{\infty} \left(\int_{|w|=r_1} \tfrac{f(w)}{w^{-k+1}}\,\mathrm{d}w \right) z^{-k}
\end{aligned}
$$

since $\left|\frac{w}{z}\right| = \frac{r_1}{|z|} < 1$ so that $w \mapsto \sum_{k=0}^{\infty} \frac{f(w)}{w}\left(\frac{w}{z}\right)^k$ converges uniformly on $\{w \in \mathbb{C} \,|\, |w| = r_1\}$ (by Lemma IV.1.4).

Then we note that

$$\int_{|w|=r_i} \frac{f(w)}{w^{k+1}}\,\mathrm{d}w = \int_{|w|=r} \frac{f(w)}{w^{k+1}}\,\mathrm{d}w$$

for any $r \in (R_1, R_2)$ by the Homotopy version of Cauchy's theorem since $w \mapsto \frac{f(w)}{w^{k+1}}$ is holomorphic in A for any $k \in \mathbb{Z}$.

To show that the representation is unique, let $f(z) = \sum_{k=-\infty}^{\infty} b_k z^k$ be a representation of f, where the convergence is in $R_1 < |z| < R_2$. Choose r such that $R_1 < r < R_2$. The series converges uniformly on $\partial B_r(0) = \{z \,|\, |z| = r\}$, so we may multiply by $1/2\pi i z^{n+1}$ and integrate around $\partial B_r(0)$. Interchanging summation and integration (using Lemma IV.1.4), the only term which survives (see Example on p. 39) is the one where $n - k + 1 = 1$ (i.e., $n = k$) and we obtain

$$a_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}}\,\mathrm{d}z = \sum_{k=-\infty}^{+\infty} b_k \frac{1}{2\pi i}\int_{|z|=r} \frac{1}{z^{n-k+1}}\,\mathrm{d}z = b_n.$$

□

---

[8] For the step from the first to the second line, observe that $\frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\,\mathrm{d}w = 0$ by Cauchy's Theorem where Γ is the grey contour in the picture to the right and note that the direction is chosen such that the paths between $\partial B_{r_1}(z_0)$ and $\partial B_{r_2}(z_0)$ cancel with $\partial B_\varrho(z)$.

V. Residue Calculus

V.1. Isolated Singularities

References: [DET, Section 4.6] and [ST, Sections 11.2 & 11.3] (Characterisation of singularities: [DET, Theorems 4.6.2 & 4.6.3 & 4.6.4] and [ST, Lemma 11.2 & Corollary 11.6 & Theorem 11.7]; Theorem of Casorati-Weierstrass: [DET, Theorem 4.6.4] and [ST, Theorem 11.7]; Picard's Theorem: [DET, Theorem 4.6.5]; meromorphic functions: [DET, Section 5.4] and [ST, Section 11.6])

Definition. We call $z_0 \in \mathbb{C}$ an *isolated singularity of* f if f is not defined at z_0 but holomorphic in some punctured neighbourhood $B^*_R(z_0) = B_R(z_0) \setminus \{z_0\}$ of z_0.

Remark. Thus we think of an isolated singularity as a point where f is *undefined*, not as a point where f is *undefinable*.

We consider the Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k \qquad \text{on} \qquad B^*_R(z_0)$$

of f at an isolated singularity and classify isolated singularities according to the behaviour of its "principal part"

$$\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-1}.$$

Definition. An isolated singularity z_0 of f is called

(i) a *removable singularity* if $f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$ on $B^*_R(z_0)$, i.e., if the principal part vanishes;

(ii) a *pole of order* $n \in \mathbb{N}$ if $f(z) = \sum_{k=-n}^{\infty} a_k (z-z_0)^k$ on $B^*_R(z_0)$ with $a_{-n} \neq 0$, i.e., the principal part is a finite sum at a pole;

(iii) an *essential singularity* otherwise, i.e., if the principal part has infinitely many nonzero terms.

Remark. If z_0 is a removable singularity of f, then we can holomorphically extended f to the disk $B_R(z_0)$ by setting $f(z_0) = a_0$ (see the following examples).

Example. First recall that

$$\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \dots \qquad \text{and}$$

$$\mathrm{e}^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \dots$$

(i) $z \mapsto \frac{\sin z}{z} = 1 - \frac{1}{6}z^2 + \frac{1}{120}z^4 - \dots$ has a removable singularity at the point $z_0 = 0$;
$z \mapsto \frac{\mathrm{e}^z - 1}{z} = 1 + \frac{1}{2}z + \frac{1}{6}z^2 + \dots$ has a removable singularity at the point $z_0 = 0$;

(ii) $z \mapsto \frac{\sin z}{z^2} = \frac{1}{z} - \frac{1}{6}z + \frac{1}{120}z^3 - \dots$ has a "simple pole" (a pole of order $n = 1$) at $z_0 = 0$;
$z \mapsto 1/z^2$ has a pole of order 2 at $z_0 = 0$;

(iii) $z \mapsto \sin\frac{1}{z} = \frac{1}{z} - \frac{1}{6z^3} + \frac{1}{120z^5} - \dots$ has an essential singularity at $z_0 = 0$;
$z \mapsto \mathrm{e}^{-1/z^2} = 1 - \frac{1}{z^2} + \frac{1}{2\,z^4} - \frac{1}{6\,z^6} + \dots$ has an essential singularity at $z_0 = 0$;

(iv) $z \mapsto \frac{1}{\sin\frac{1}{z}}$ has singularities for $z = \frac{1}{k\pi}$, $k \in \mathbb{Z}$ and $z = 0$; therefore $z_0 = 0$ is *not* an isolated singularity and the above classification does not apply to $z_0 = 0$.

We use the "colouring method" introduced in Section II.4 to depict these functions on the next two pages.

Remark. (Not examinable!) As already noted in the introduction on p. 5, Riemann's Zeta Function has a simple pole at $z_0 = 1$. This can be used (as one of many possibilities) to show that there are infinitely many primes, see M. Aigner and G.M. Ziegler: Proofs from THE BOOK; Springer, Berlin (1998); library[1]: `510.36 AIG`.

ⓔⓧ Let p be a polynomial. Show: $|p(z)| \to \infty$ as $|z| \to \infty$.

(Not examinable!) Regarding p as a function from the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to $\mathbb{C}$, what are we showing here in terms of singularities? Do other entire functions like exp or sin also have this property?

ⓔⓧ Prove that $z_0 \in \mathbb{C}$ is a pole of order $n \in \mathbb{N}$ of f iff there is a holomorphic function $g : B_R(z_0) \to \mathbb{C}$, defined on some disk about z_0, with $g(z_0) \neq 0$ and $g(z) = (z - z_0)^n f(z)$ for $0 < |z - z_0| < R$.

ⓔⓧ Let $f : D \to \mathbb{C}$ be holomorphic. We say that $z_0 \in D$ is a *zero of order* $m \in \mathbb{N}$ of f if the Taylor series expansion of f at z_0

$$f(z) = \sum_{k=m}^{\infty} a_k (z - z_0)^k, \qquad \text{where} \qquad a_m \neq 0.$$

Prove that $z_0 \in D$ is a zero of order $m \in \mathbb{N}$ iff there is a holomorphic function g with $g(z_0) \neq 0$ so that $f(z) = (z - z_0)^m g(z)$.

Conclude that the zeros of a nonzero holomorphic function are isolated.

ⓔⓧ Each of the following functions f has an isolated singularity at $z = 0$. Determine its nature; if it is a removable singularity define $f(0)$ so that f is holomorphic at $z = 0$; if it is a pole find the singular part; if it is an essential singularity just state it.

(i) $f(z) = \frac{\cos(z) - 1}{z}$

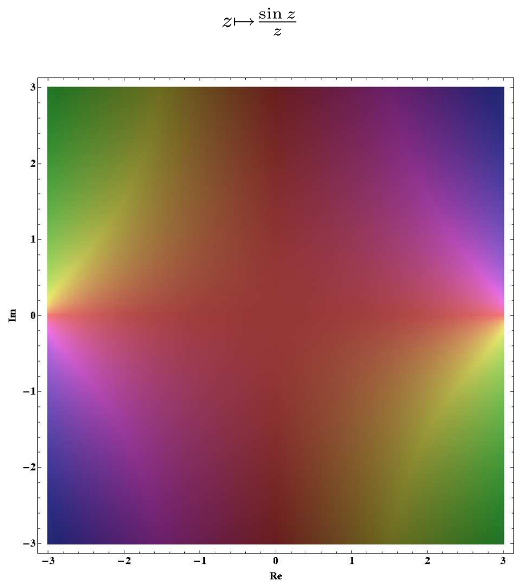
(ii) $f(z) = \mathrm{e}^{1/z}$

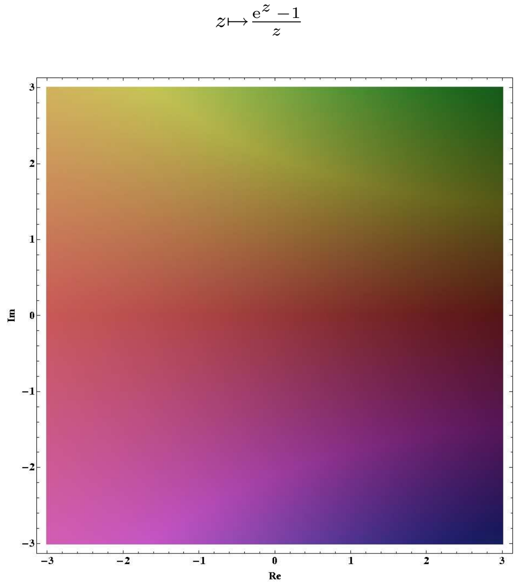
(iii) $f(z) = \frac{\cos(1/z)}{1/z}$

(iv) $f(z) = \frac{1}{1 - \mathrm{e}^z}$

---

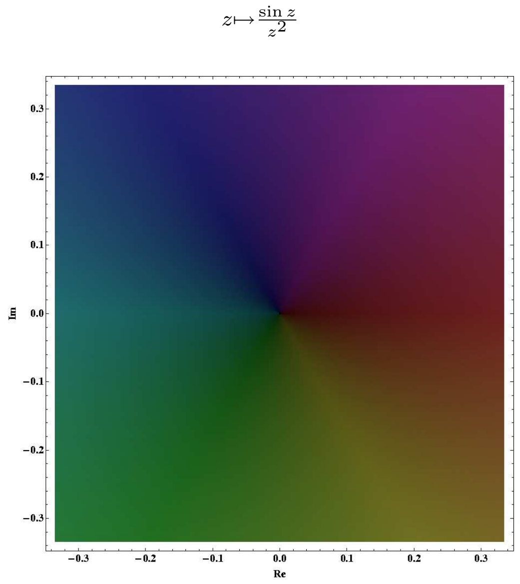
[1] The `e-book`-link of the library leads to the Italian edition of that book!

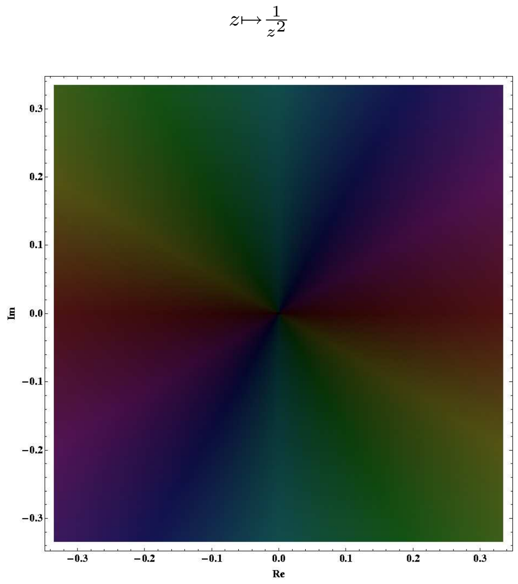




The map $z \mapsto \frac{\sin z}{z}$ has a removable singularity at 0. One can holomorphically extend this function to an entire function with value 1 at 0. Indeed, the neighbourhood of 0 is coloured red.

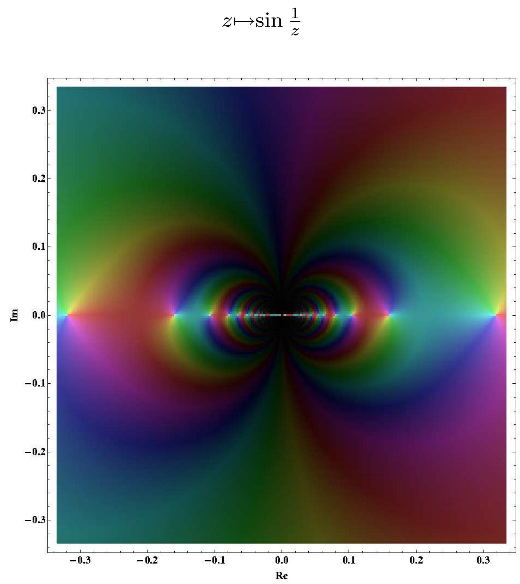
The map $z \mapsto \frac{e^z - 1}{z}$ has a removable singularity at 0. One can holomorphically extend this function to an entire function with value 1 at 0. Indeed, the neighbourhood of 0 is coloured red.



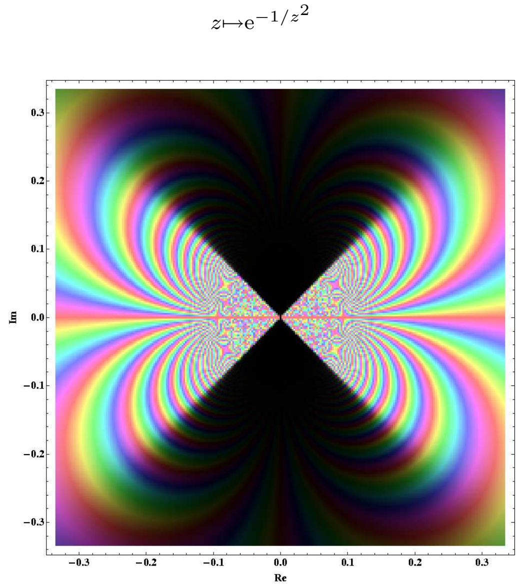


The map $z \mapsto \frac{\sin z}{z^2}$ has a pole at 0. Colours in the neighbourhood of 0 are dark.

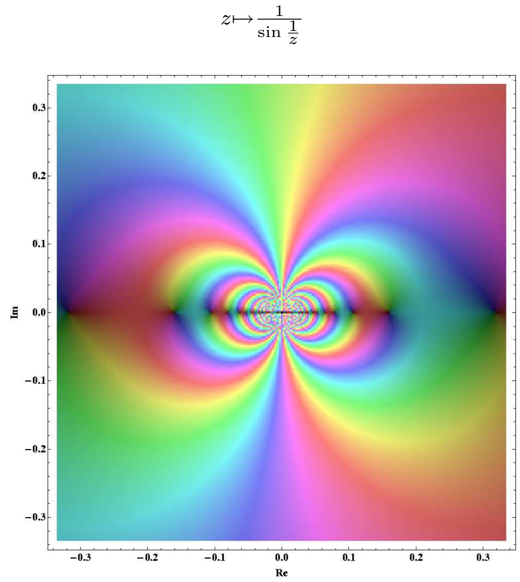
The map $z \mapsto \frac{1}{z^2}$ has a pole at 0. Colours in the neighbourhood of 0 are dark.



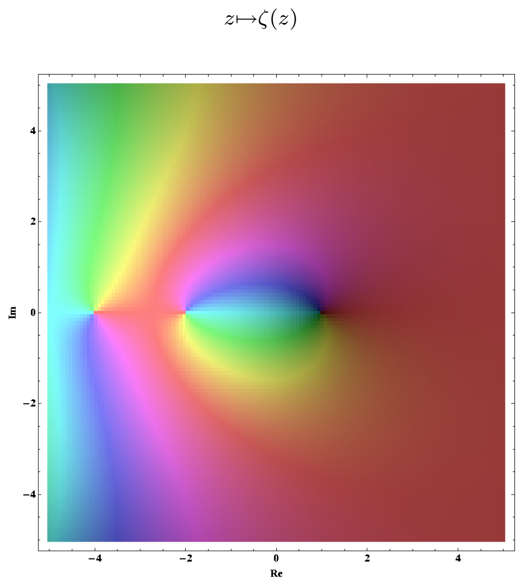
The map $z \mapsto \sin \frac{1}{z}$ has an essential singularity at 0. In the neighbourhood of 0, colours from the imaginary side(s) are dark, while they are bright along the real axis.



The map $z \mapsto \mathrm{e}^{-1/z^2}$ has an essential singularity at 0. In the neighbourhood of 0, bright (along the imaginary axis) and dark (along the real axis) can be found. Compare this complex function with the corresponding (very smooth) real function on p. 67.



The map $z \mapsto 1/\sin \frac{1}{z}$ has no isolated singularity at 0. Poles can be found at $z = \frac{1}{k\pi}$, $k \in \mathbb{Z}$, which is reflected in the black spots along the real axis (with an accumulation point at 0).



Riemann's Zeta Function has a simple pole at $z_0 = 1$ and is holomorphic on $\mathbb{C} \setminus \{1\}$, compare Section IV.5. This can be used to show that there are infinitely many primes.

Theorem V.1.1 (Characterization of singularities)**.** *Suppose $z_0 \in \mathbb{C}$ is an isolated singularity of f, i.e., $f : B_R^*(z_0) \to \mathbb{C}$ is holomorphic. Then*

- (i) *z_0 is removable iff $\limsup_{z\to z_0} |f(z)| < \infty$, i.e., iff f is bounded in a neighbourhood of z_0;*
- (ii) *z_0 is a pole iff $|f(z)| \to \infty$ as $z \to z_0$;*
- (iii) *z_0 is an essential singularity iff, for every $c \in \mathbb{C}$, there is a sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \to z_0$ and $f(z_n) \to c$ for $n \to \infty$.*

Remark. Part (iii) is the remarkable part – its essence can be rephrased as the

Theorem (Theorem of Casorati-Weierstrass)**.** *In any neighbourhood of an essential singularity z_0 of f the values $f(z)$ get arbitrarily close to any given value $c \in \mathbb{C}$.*

In fact, more is true and we state without proof (cf. [DET, Theorem 4.6.5]):

Theorem (Picard's Theorem)**.** *If $f(z)$ has an essential singularity at z_0, then it takes on every but possibly one value in every punctured disk around z_0, i.e., the set $\mathbb{C} \setminus \{f(z) \mid z \in B_\varepsilon^*(z_0)\}$ is at most a singleton.*

Note that we therefore have a sequence $z_n \to z_0$ so that $f(z_n) = c$ for every but possibly one given $c \in \mathbb{C}$. Compare this statement with the Identity Theorem for holomorphic functions (Theorem IV.3.2)!

Proof. . . . of Theorem V.1.1.

(i), ⇒: If z_0 is removable then f has a holomorphic and, in particular, continuous extension (also called f) to $B_R(z_0)$. Hence f is bounded on any disk $\overline{B}_r(z_0)$ with $r < R$.

(i), ⇐: Suppose $|f(z)| \leq M$ for $0 < |z - z_0| \leq r$ for some $r < R$. Then, for $n \in \mathbb{N}$ and any $0 < \varrho \leq r$, the ML-inequality yields

$$|a_{-n}| = \frac{1}{2\pi}\left|\int_{|w-z_0|=\varrho} f(w)(w-z_0)^{n-1}\,\mathrm{d}w\right| \leq M\varrho^n.$$

Hence $a_{-n} = 0$ since $0 < \varrho \leq r$ is arbitrary.

(ii), ⇒: If z_0 is a pole of order n then $g(z) = (z-z_0)^n f(z)$ is holomorphic (more accurately: has a holomorphic extension also called g) in $B_R(z_0)$ with $a_{-n} = g(z_0) \neq 0$. By continuity of g at z_0

$$\exists \delta > 0\, \forall z \in B_\delta(z_0) : \frac{1}{2}|g(z_0)| < |g(z)| < \frac{3}{2}|g(z_0)|.$$

Hence $|f(z)| \to \infty$ since $\frac{|g(z_0)|}{2|z-z_0|^n} \to \infty$ as $z \to z_0$.

(ii), ⇐: Suppose $|f(z)| \to \infty$ as $z \to z_0$. In particular,

$$\exists r > 0\, \forall z \in B_r(z_0) : |f(z)| > 1$$

so that

$$g : B_r^*(z_0) \to \mathbb{C}, \quad z \mapsto g(z) = \frac{1}{f(z)}$$

is a well defined holomorphic function which is bounded and therefore, by (i), extends holomorphically to $B_r(z_0)$ with $g(z_0) = 0$.

Since $g \not\equiv 0$ we have, for $z \in B_r(z_0)$,

$$g(z) = \sum_{k=n}^{\infty} b_k (z - z_0)^k = (z - z_0)^n h(z)$$

where h is holomorphic on $B_r(z_0)$ with $b_n = h(z_0) \neq 0$ for some $n \in \mathbb{N}$.

Thus $h(z) \neq 0$ for $z \in B_\delta(z_0)$ for some $\delta > 0$ (by continuity), and

$$f(z) = \frac{1}{g(z)} = \frac{1}{(z - z_0)^n} \frac{1}{h(z)} = \frac{1}{(z - z_0)^n} \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

has a pole (of order $n \in \mathbb{N}$) at z_0.

(iii), ⇒: This is the Casorati-Weierstrass Theorem (Proof (ex)).

(iii), ⇐: By the assumption f is neither bounded near z_0 nor do we have $|f(z)| \to \infty$ as $z \to z_0$. Therefore, by (i) & (ii), z_0 is neither a removable singularity nor a pole. Hence it must be an essential singularity.

$\square$

(ex) Prove that g has a pole of order $n \in \mathbb{N}$ at z_0 if and only if $f(z) = \frac{1}{g(z)}$ has a zero of order n.

(ex) Prove the Casorati-Weierstrass Theorem.
Hint: Proof by contradiction; fix c and consider $g = \frac{1}{f - c}$.

Definition. A function f is called *meromorphic in D* if it is holomorphic in D except for poles.

Example. The function $z \mapsto f(z) = \frac{1}{\sin z}$ is meromorphic in $\mathbb{C}$: it is holomorphic in $\mathbb{C} \setminus \{k\pi \,|\, k \in \mathbb{Z}\}$ and has simple poles at $z = k\pi$.

(ex) Convince yourself that $z \mapsto \frac{1}{\sin z}$ is meromorphic in $\mathbb{C}$.

(ex) Suppose the $f, g : D \to \mathbb{C}$ are holomorphic and $g \not\equiv 0$. Prove that $\frac{f}{g}$ is meromorphic in D.

V.2. The Residue Theorem

References: [DET, Section 5.1] and [ST, Chapter 12]
(Residue Theorem: [DET, Theorem 5.1.1] and [ST, Theorem 12.1])

Definition. Suppose $f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$ in $B_R^*(z_0)$ for some $R > 0$. Then the coefficient $a_{-1} = \operatorname{Res}(f, z_0)$ is called the *residue of f at z_0*.

Remark. By Laurent's Theorem

$$\operatorname{Res}(f, z_0) = a_{-1} = \frac{1}{2\pi i} \int_{|w-z_0|=r} f(w)\,\mathrm{d}w$$

for any $r < R$. In particular, if z_0 is a removable singularity of f, i.e., f extends holomorphically to $B_R(z_0)$, then $\operatorname{Res}(f, z_0) = 0$.

Remark. If f has a simple pole at z_0 then

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

We have $f(z) = \frac{a_{-1}}{z-z_0} + g(z)$ with g holomorphic in some disk $B_R(z_0)$, so that

$$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} (a_{-1} + (z - z_0) g(z)) = a_{-1} + 0 \cdot g(z_0) = \operatorname{Res}(f, z_0).$$

ⓔⓧ Prove the p/q'-rule: Suppose $p, q : B_R(z_0) \to \mathbb{C}$ are holomorphic and q has a zero of order $n = 1$ at z_0, i.e., $q(z_0) = 0$ and $q'(z_0) \neq 0$. Then $f = \frac{p}{q}$ has $\operatorname{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}$.

ⓔⓧ Let z_0 be a pole of order n of f, i.e., $f(z) = \frac{g(z)}{(z-z_0)^n}$, where g is holomorphic in some $B_R(z_0)$ with $g(z_0) \neq 0$. Prove that

$$\operatorname{Res}(f, z_0) = \frac{g^{(n-1)}(z_0)}{(n-1)!}.$$

Hint: Use Cauchy's formulae for the derivatives.

Theorem V.2.1 (Residue Theorem). *Let f be meromorphic in a domain D and suppose $\Gamma \subset D$ is a simple closed contour with interior $I_\Gamma \subset D$ and no poles of f on Γ. Then*

$$\int_\Gamma f\,\mathrm{d}z = 2\pi i \sum_{z \in I_\Gamma} \operatorname{Res}(f, z).$$

Remark. (i) If D is simply connected then the condition $I_\Gamma \subset D$ is automatically fulfilled for any simple closed contour $\Gamma \subset D$.

(ii) The condition on Γ of being a *simple* closed contour can be relaxed; then the above formula has to be modified by introducing winding numbers as fore-factors.

(iii) The apparently infinite sum on the right hand side of the above formula is, in fact, finite so that the formula makes sense.

Proof. First note that the sum $\sum_{z \in I_\Gamma} \operatorname{Res}(f, z)$ is a finite sum. We show this by contraposition.

By the Jordan Curve Theorem, $\Gamma \cup I_\Gamma \subset \mathbb{C}$ is compact. Now suppose that I_Γ contained infinitely many poles of f. Then, by the Bolzano-Weierstrass Theorem, there would be a limit/accumulation point $z_0 \in D$ of the poles. Now, f can neither be differentiable at z_0 nor can z_0 be an isolated singularity, in particular, not a pole (wherefore such a function f is not meromorphic). Consequently, there are only finitely many poles $z_1, \ldots, z_m$ of f in $I_\Gamma \subset \Gamma \cup I_\Gamma$.

At all other points, $z \neq z_1, \dots, z_m$, f is holomorphic and therefore $\operatorname{Res}(f,z) = 0$ so that

$$\sum_{z \in I_\Gamma} \operatorname{Res}(f,z) = \sum_{i=1}^{m} \operatorname{Res}(f,z_i).$$

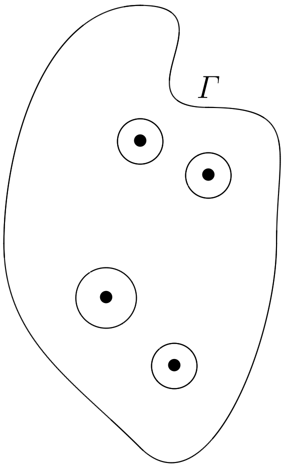
Now, since the z_i are isolated there are $\varrho_i > 0$ so that

$$\overline{B}_{\varrho_i}(z_i) \cap \overline{B}_{\varrho_j}(z_j) = \varnothing \qquad \text{for} \qquad i \neq j.$$

Then

$$\int_\Gamma f \, \mathrm{d}z = \sum_{i=1}^{m} \int_{|z-z_k|=\varrho_k} f \, \mathrm{d}z = 2\pi i \sum_{i=1}^{m} \operatorname{Res}(f,z_i)$$

by the Homotopy version of Cauchy's Theorem.



□

V.3. Evaluation of Real Integrals

References: [DET, Section 5.2] and [ST, Sections 12.3]

One important application of the Residue Theorem is the evaluation of real integrals that cannot be evaluated by more elementary means.

These are often of the form

$$\int_0^\infty f(x) \, \mathrm{d}x \qquad \text{or} \qquad \int_{-\infty}^{\infty} f(x) \, \mathrm{d}x.$$

In the absence of Lebesgue's theory of integration we interpret these as improper Riemann integrals:

Recall (from M11).

- $\int_0^\infty f(x) \, \mathrm{d}x = \lim_{R\to\infty} \int_0^R f(x) \, \mathrm{d}x$ if the limit exists, and
- $\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = \lim_{R\to\infty} \int_0^R f(x) \, \mathrm{d}x + \lim_{r\to-\infty} \int_r^0 f(x) \, \mathrm{d}x$ and both limits are required to exist.

Sometimes the latter is replaced by the "principal value integral":

Definition. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.

$$\mathrm{PV} \int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x) \, \mathrm{d}x$$

is called the *principal value integral of* f if the limit exists.

Remark. The improper integral $\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x$ might not exist even though the principal value integral $\mathrm{PV} \int_{-\infty}^{\infty} f(x) \, \mathrm{d}x$ does:

$$\int_{-r}^{R} x \, \mathrm{d}x = \frac{R^2}{2} - \frac{r^2}{2}$$

has no limit as $R, r \to \infty$ but the limit $r = R \to \infty$ does exist.

However, if $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ does exist, then the principal value integral exists and

$$\mathrm{PV}\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = \int_{-\infty}^{\infty} f(x)\ \mathrm{d}x.$$

Example. Integrate $\int_0^\infty \frac{\mathrm{d}x}{1+x^4}$.

(i) *We associate with the given real integral a related contour integral, of the form $\int_\Gamma f(z)\ \mathrm{d}z$.*

We observe that

$$2\int_0^R \frac{\mathrm{d}x}{1+x^4} = \int_{-R}^R \frac{\mathrm{d}x}{1+x^4}.$$

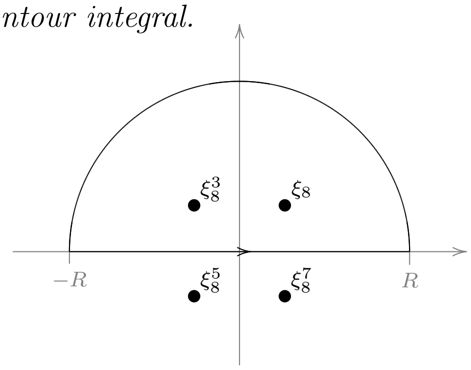
So, we consider the contour integral $\int_\Gamma \frac{1}{1+z^4}\ \mathrm{d}z$ where $\Gamma = [-R, R] \cup \Gamma(R)$ (with $\Gamma(R) = \{R\,\mathrm{e}^{it}\,|t \in [0,\pi]\}$) is a semicircular contour.

(ii) *We use the Residue Theorem to evaluate the contour integral.*

Note that we have

$$f(z) = \frac{1}{1+z^4}$$
$$= \frac{1}{(z-\xi_8)(z-\xi_8^3)(z-\xi_8^5)(z-\xi_8^7)}$$

where $\xi_8 = \mathrm{e}^{i\pi/4}$. Thus f has simple poles at ξ_8^k $(k = 1, 3, 5, 7)$, and we calculate the residue



$$\mathrm{Res}(f, \xi_8^k) = \begin{cases} \lim\limits_{z\to\xi_8^k} (z-\xi_8^k)\, f(z), & \text{using Remark on p. 86}, \\ \frac{1}{4\,\xi_8^{3\,k}}, & \text{using the } p/q'\text{-rule (p. 86)}, \end{cases}$$
$$= -\frac{1}{4}\,\xi_8^k, \qquad \text{since } \left(\xi_8^k\right)^4 = -1.$$

Then the Residue Theorem yields (for $R > 1$)

$$\int_\Gamma \frac{1}{1+z^4}\ \mathrm{d}z = 2\pi i\ \left(\mathrm{Res}(f,\xi_8) + \mathrm{Res}(f,\xi_8^3)\right)$$
$$= 2\pi i\left(-\frac{1}{4}\right)\cdot\left(\xi_8 + \xi_8^3\right) = -\frac{1}{2}\pi i \cdot \sqrt{2} i = \frac{\sqrt{2}}{2}\,\pi.$$

(iii) *We split the contour integral into two parts: a real integral we are interested in, and a complex integral we want to get rid of.*

Obviously,

$$\int_\Gamma \frac{1}{1+z^4}\ \mathrm{d}z = \int_{-R}^R \frac{1}{1+x^4}\ \mathrm{d}x + \int_{\Gamma(R)} \frac{1}{1+z^4}\ \mathrm{d}z = \int_{-R}^R \frac{1}{1+x^4}\ \mathrm{d}x + \int_0^\pi \frac{iR\,\mathrm{e}^{it}}{1+R^4\,\mathrm{e}^{4it}}\ \mathrm{d}t.$$

(iv) *We use the ML-inequality to show that this complex integral becomes arbitrarily small in modulus if we take R to be large enough.*

Using the ML-inequality, we have (by $|z^4+1| \geq |z|^4 - 1$)

$$\left|\int_{\Gamma(R)} \frac{1}{1+z^4}\,\mathrm{d}z\right| \leq \left(\max_{t\in[0,\pi]}\left|\frac{1}{1+R^4\,\mathrm{e}^{4it}}\right|\right)\cdot \pi\,R$$
$$\leq \frac{1}{R^4-1}\cdot \pi R \;=\; \pi\,\frac{R}{R^4-1} \;\to\; 0 \quad \text{as } R\to\infty.$$

Hence the integral $\int_0^\infty \frac{\mathrm{d}x}{1+x^4}$ exists and

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^4} = \frac{1}{2}\,\mathrm{PV}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{1+x^4} = \pi i\,\left(\mathrm{Res}(f,\xi_8)+\mathrm{Res}(f,\xi_8^3)\right) = \frac{\pi}{2\sqrt{2}}.$$

(ex) Evaluate $\int_0^\infty \frac{\cos x\,\mathrm{d}x}{(1+x^2)(4+x^2)}$ using the Theorem of Residues.
Hint: Consider $f(z) = \frac{\mathrm{e}^{iz}}{(z^2+1)(z^2+4)}$.

(ex) Compute $\int_{|z|=2}\frac{\mathrm{d}z}{z^3-1}$.

(ex) Let $\alpha > 0$. Show that $\int_{-\infty}^{\infty}\frac{\cos \alpha x}{1+x^2}\,\mathrm{d}x = \pi\,\mathrm{e}^{-\alpha}$.
Remark: This is a question from the 2006 exam.

(ex) For $n \in \mathbb{N}$, evaluate

$$\int_{|z|=1}\frac{\mathrm{e}^{(z^n)}}{z}\,\mathrm{d}z.$$

Hence show that

$$\int_0^{2\pi}\mathrm{e}^{\cos(n\vartheta)}\,\cos\left(\sin(n\vartheta)\right)\mathrm{d}\vartheta = 2\pi$$

and

$$\int_0^{2\pi}\mathrm{e}^{\cos(n\vartheta)}\,\sin\left(\sin(n\vartheta)\right)\mathrm{d}\vartheta = 0.$$

Remark: This is a question from the 2008 exam.

(ex) Evaluate $\int_0^\infty \frac{\sin x\,\mathrm{d}x}{x}$.
Remark: This is a hard one!

V.4. Evaluation of Infinite Sums

REFERENCES: [DET, Section 5.1] and [ST, Chapter 12]
(Computation of $\zeta(2)$: [DET, Example 5.1.3] and [ST, Section 12.4])

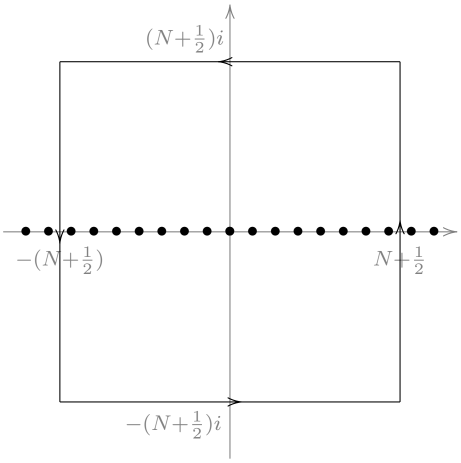
Example. We wish to compute $\zeta(2) = \sum_{n=1}^\infty \frac{1}{n^2}$.

Consider $f(z) = \frac{\pi \cot \pi z}{z^2}$. We intend to apply the Residue Theorem to the integral

$$\int_{\max\{|\operatorname{Re} z|, |\operatorname{Im} z|\} = N+\frac{1}{2}} f \, \mathrm{d}z$$

and then take the limit $N \to \infty$.

The "trick" here is that this integral vanishes for $N \to \infty$ wherefore the sum over all residues equals 0. One then has to relate that sum with the infinite sum we are interested in.



First we determine the residues of f:

(i) The function $g(z) = \pi \cot(\pi z) = \pi \cdot \frac{\cos(\pi z)}{\sin(\pi z)}$ is holomorphic for all $z \notin \mathbb{Z}$ and periodic in $\mathbb{Z}$ (i.e., $g(z+n) = g(z)$ for $n \in \mathbb{Z}$). Its Laurent-series on $B^*_\pi(0)$ is given by[2]

$$g(z) = \frac{1}{z} - \frac{\pi^2}{3} z - \frac{\pi^4}{45} z^3 - \frac{2\pi^6}{945} z^5 - \dots$$

So, g has simple poles of residue 1 at every integer.

(ii) Then $f(z) = g(z)/z^2$ has simple poles for $z = n$, $n \in \mathbb{Z} \setminus \{0\}$, and a triple pole at $z = 0$. Thus

$$\operatorname{Res}(f, n) = \lim_{h \to 0} h f(n+h) = \lim_{h \to 0} \frac{h \cdot g(n+h)}{(n+h)^2} = \lim_{h \to 0} \frac{h \cdot g(h)}{(n+h)^2} = \frac{1}{n^2}$$

for $n \neq 0$ by the periodicity of g. Moreover, the above Laurent-series for g also yields $\operatorname{Res}(f, 0) = -\frac{\pi^2}{3}$.

Now we use the ML-inequality for the above contour integral: Obviously, the length of the contour is $L = 8 \cdot (N + \frac{1}{2})$. For M we observe[3]:

(i) $g(x+iy) = \pi \, \frac{\cos(\pi x)\, \sin(\pi x) \; - \; i \cdot \cosh(\pi y)\, \sinh(\pi y)}{\cosh^2(\pi y)\, \sin^2(\pi x) + \cos^2(\pi x)\, \sinh^2(\pi y)}$

(ii) we have for the "vertical parts" of the contour

$$\left| g\left(iy \pm \left(N + \frac{1}{2} \right) \right) \right| = \left| g\left(\frac{1}{2} + iy \right) \right| = \pi \left| \frac{\sinh(\pi y)}{\cosh(\pi y)} \right| \leq \pi .$$

(iii) we have for the "horizontal parts" of the contour (using $|\cos x|, |\sin x| \leq 1$, $\sinh y < \cosh y$ etc.)

$$\left| g\left(x \pm i \left(N + \frac{1}{2} \right) \right) \right| \leq \pi \, \frac{1 + \cosh\left(\pi \left(N + \frac{1}{2}\right)\right) \sinh\left(\pi \left(N + \frac{1}{2}\right)\right)}{\sinh^2\left(\pi \left(N + \frac{1}{2}\right)\right)} \overset{N \geq 1}{\leq} 1.28\, \pi$$

---

[2] To obtain the Laurent series, we observe that $\pi z \cdot \cot(\pi z) = \frac{\pi z}{\sin(\pi z)} \cdot \cos(\pi z)$ has a removable singularity at 0. Using the Taylor series for sine and cosine in the relationship $\pi z \cdot \cot(\pi z) \cdot \sin(\pi z) = \pi z \cdot \cos(\pi z)$, one can sucessively obain the coefficients of the Laurent-series for the cotanget.

[3] Using $\sin(x+iy) = \sin x \, \cosh y + i \, \cos x \, \sinh y$ and $\cos(x+iy) = \cos x \, \cosh y - i \, \sin x \, \sinh y$.

Therefore

$$|f(z)| \le \frac{|g(z)|}{|z|^2} \le \frac{1.28\,\pi}{(N+\frac{1}{2})^2}$$

for $z \in \mathit{\Gamma}_N = \{x+iy \mid \max\{|x|,|y|\} = N+\frac{1}{2}\}$. The ML-inequality then gives

$$\left|\int_{\mathit{\Gamma}_N} f\,\mathrm{d}z\right| \le \frac{1.28\,\pi}{(N+\frac{1}{2})^2}\cdot 8 \cdot \left(N+\frac{1}{2}\right) \longrightarrow 0 \qquad \text{as} \qquad N \longrightarrow \infty.$$

Thus the Residue Theorem yields

$$-\frac{\pi^2}{3} + 2\sum_{k=1}^{N}\frac{1}{k^2} = \frac{1}{2\pi i}\int_{\max\{|\operatorname{Re} z|,|\operatorname{Im} z|\}=N+\frac{1}{2}} f\,\mathrm{d}z \longrightarrow 0$$

as $N \longrightarrow \infty$, that is,

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

Remark. Similar computations yield

$$\zeta(4) = \sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}, \qquad\qquad \zeta(6) = \sum_{k=1}^{\infty}\frac{1}{k^6} = \frac{\pi^6}{945},$$

and, ex using the function $\frac{\pi}{z^2\,\sin(\pi z)}$, also $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$.

Remark. (Not examinable!) One can use the functional equation of Riemann's Zeta Function (compare p. 75) to calculate its values at the (odd) negative integers: Since

$$\zeta(z) = -2^z\,\pi^{z-1}\,z\,\sin\left(\frac{1}{2}\pi\,z\right)\,\Gamma(-z)\,\zeta(1-z)$$

we have[4]

$$\zeta(-1) = -\frac{1}{2\,\pi^2}\cdot\zeta(2) = -\frac{1}{12}.$$

Remark. Also observe that one cannot use the above method to calculate $\zeta(3) = \sum_{k=1}^{\infty}\frac{1}{k^3}$, $\zeta(5) = \sum_{k=1}^{\infty}\frac{1}{k^5}$ etc.; in fact, little is know about these values of Riemann's Zeta Function. Note that $\zeta(3)$ is also known as *Apéry's Constant*, but that is almost all one knows about it, see

http://en.wikipedia.org/wiki/Apéry's_constant
http://mathworld.wolfram.com/AperysConstant.html

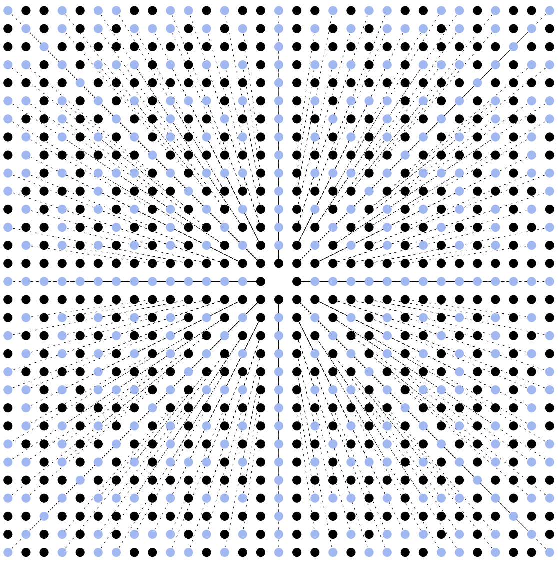
Remark. The above method is, of course, not the only possibility to establish that $\zeta(2) = \frac{\pi^2}{6}$. It is the one usually used in lectures on complex analysis. For a list of more proofs (the above one is listed as "Proof 9"), see
http://www.secamlocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf

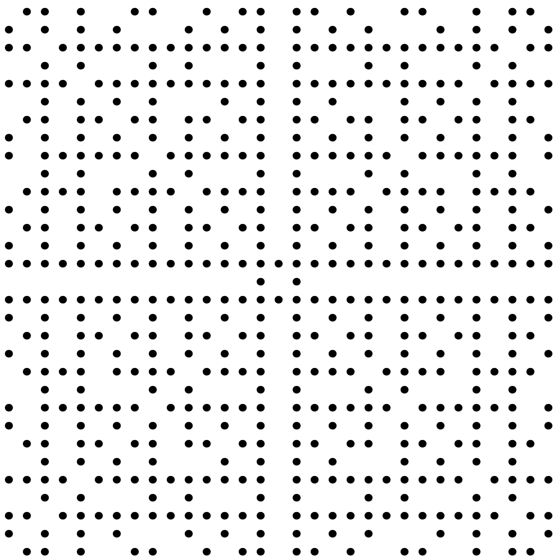
---

[4] However, note that *clearly* we do *not* have "$\sum_{n=1}^{\infty}\frac{1}{n^{-1}} = 1+2+3+4+5+\ldots = -\frac{1}{12}$". At $z=-1$, we are talking about the analytic continuation of ζ, the definition via the infinite sum does not make sense there (it is not convergent).

Remark. Riemann' Zeta Function is closely related to properties of primes and natural numbers ("Riemann Hypothesis"), e.g., $1/\zeta(2) \approx 0.6079$ is the probability that two arbitrarily chosen natural numbers are relatively prime.

A nice interpretation of this number uses *Euclid's Orchard*: Suppose you stand in "the middle" of a forest where the trees are planted on the square lattice $\mathbb{Z}^2 \setminus \{0\}$ (well, you are standing on the origin). What fraction of trees do you see? You can convince yourself that a tree/lattice point is *visible* from the origin iff its coordinates are relatively prime. Thus, one can see almost 61% of all trees from the origin!





This concludes our exploration of Complex Analysis. . .

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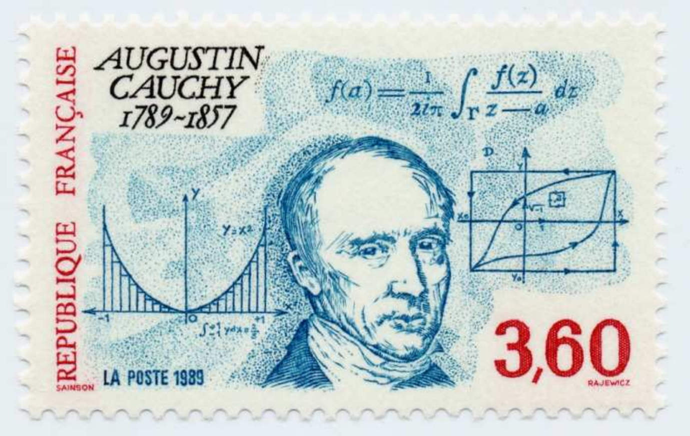

Good wishes:

> for your further studies or for whatever your plans for the future are;
>
> and, of course, for the upcoming exam.

I hope that I was able to explain some of the basic ideas and concepts of Complex Analysis to you and that I was able to share with you some of the excitement about the many results, which do not hold in the 'real' world of MA7/11, but require the complex realm.

Bernd Sing

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